First formulation of SLAM

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What is SLAM?

- **Localization**: estimating the robot’s location
- **Mapping**: building a map
- **SLAM**: computing the robot’s pose and the environment map simultaneously
Definition of the SLAM problem

• Input:
  • Robot’s controls
    • \( u_{1:T} = \{u_1, u_2, \ldots, u_T\} \)
  • Observations
    • \( z_{1:T} = \{z_1, z_2, \ldots, z_T\} \)

• Output
  • Poses of the robots
    • \( x_{0:T} = \{x_0, x_1, x_2, \ldots, x_T\} \)
  • Map of the environment
    • \( m \)
Map representations

• Grid-based
  • Occupancy grid with typically fixed resolution

• Landmark-based
  • The map consists of a set of isolated landmarks
  • A landmark is described, e.g., by a pose location wrt a frame
Landmark-based SLAM

• The robot learns the locations of the landmarks while localizing itself
• State variables
  • Robot pose
  • Coordinates of each of the landmarks
• The problem involves different aspects
  • Landmark extraction
  • Data association
  • State estimation
  • State update
  • Landmark update
First formulation of SLAM

- Smith et al. [1990] present
  - *Stochastic map*: representation for spatial relationships between objects
  - A set of procedures for
    - Reading information from it
    - Building/updating it
Spatial Relationship

• A *spatial relationship* is represented by the vector of its *spatial variables*: e.g., the position and orientation of one, in the frame of reference of the other in 2D.

\[
\mathbf{x}_1 = \mathbf{x} = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \end{bmatrix}
\]
Uncertain spatial relationship in 2D

• An *uncertain spatial relationship* is represented by a *probability distribution* over its spatial variables, e.g., with a mean and a covariance matrix.

\[
\hat{x} \triangleq E(x) \quad \text{C}(x) \triangleq E((x - \hat{x})(x - \hat{x})^T)
\]

\[
\hat{x} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{\phi} \end{bmatrix}, \quad \text{C}(x) = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{x\phi} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{y\phi} \\ \sigma_{x\phi} & \sigma_{y\phi} & \sigma_{\phi}^2 \end{bmatrix}
\]
A stochastic map models $n$ uncertain spatial relationships with the system state vector (all spatial variables wrt world reference frame) and with the associated system covariance matrix

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix} \quad C(x) = \begin{bmatrix} C(x_1) & C(x_1, x_2) & \cdots & C(x_1, x_n) \\ C(x_2, x_1) & C(x_2) & \cdots & C(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ C(x_n, x_1) & C(x_n, x_2) & \cdots & C(x_n) \end{bmatrix}
$$

where $C(x_i, x_j) \triangleq E((x_i - \hat{x}_i)(x_j - \hat{x}_j)^\top)$

$C(x_j, x_i) = C(x_i, x_j)^\top$
Estimating the first two moments of unknown multivariate probability distributions

- Consider the non-linear mapping \( y = f(x) \)
- approximate using Taylor Series
  \[
  y = f(\hat{x}) + F_x(x - \hat{x}) + \cdots,
  \]
  where
  \[
  F_x \triangleq \frac{\partial f(x)}{\partial x}(\hat{x}) \triangleq \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
  \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{\partial f_r}{\partial x_1} & \frac{\partial f_r}{\partial x_2} & \cdots & \frac{\partial f_r}{\partial x_n}
  \end{bmatrix}_{x=\hat{x}}.
  \]

  - The first-order estimate of the mean: \( \hat{y} \approx f(\hat{x}) \).

  And the first-order estimate of the covariances:
  \[
  C(y) \approx F_x C(x) F_x^T,
  
  C(y, z) \approx F_x C(x, z),
  
  C(z, y) \approx C(z, x) F_x^T.
  \]
How to read from the map

• In a real system, it is useful to get the information from the stochastic map with respect to a different frame than the world frame
  • e.g., motion of the robot or its observations with respect to the robot’s frame

• Estimate the resultant relationship between initial and final frames
  • Compounding operation
  • Reversal operation
Compounding operation

- For example, given $0_2$ and $x_{23}$ how do we compute the resultant relationship $x_{03}$?
Compounding operation

• Given two spatial relationships $x_{ij}$ and $x_{jk}$, calculate the resultant relationship $x_{ik}$

$$x_{ik} \triangleq x_{ij} \oplus x_{jk}$$

$$= \begin{bmatrix}
x_{jk} \cos \phi_{ij} - y_{jk} \sin \phi_{ij} + x_{ij} \\
x_{jk} \sin \phi_{ij} + y_{jk} \cos \phi_{ij} + y_{ij} \\
\phi_{ij} + \phi_{jk}
\end{bmatrix}$$
Compounding operation

• The first-order estimate of the mean of the compounding operation is

\[ \hat{x}_{ik} \approx \hat{x}_{ij} \oplus \hat{x}_{jk} \]

• And the first-order estimate of the covariance is

\[
C(x_{ik}) \approx J_\oplus \begin{bmatrix}
C(x_{ij}) & C(x_{ij}, x_{jk}) \\
C(x_{jk}, x_{ij}) & C(x_{jk})
\end{bmatrix} J_\oplus^T
\]

where

\[
J_\oplus \triangleq \frac{\partial (x_{ij} \oplus x_{jk})}{\partial (x_{ij}, x_{jk})} = \frac{\partial x_{ik}}{\partial (x_{ij}, x_{jk})} = \begin{bmatrix}
1 & 0 & -(y_{ik} - y_{ij}) & \cos \phi_{ij} & -\sin \phi_{ij} & 0 \\
0 & 1 & (x_{ik} - x_{ij}) & \sin \phi_{ij} & \cos \phi_{ij} & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]
Reversal operation

• For example, how to compute $x_{21}$? We need first $x_{20}$
Reversal operation

• Given $x_{ij}$, calculate $x_{ji}$

$$x_{ji} \triangleq \Theta x_{ij} \triangleq \begin{bmatrix} -x_{ij} \cos \phi_{ij} - y_{ij} \sin \phi_{ij} \\ x_{ij} \sin \phi_{ij} - y_{ij} \cos \phi_{ij} \\ -\phi_{ij} \end{bmatrix}$$

• The estimate of the mean

$$\hat{x}_{ji} = \Theta \hat{x}_{ij}$$

• The estimate of the covariances

$$C(x_{ji}) \approx J_\Theta C(x_{ij}) J^T_\Theta$$

$$J_\Theta \triangleq \frac{\partial x_{ji}}{\partial x_{ij}} = \begin{bmatrix} -\cos \phi_{ij} & -\sin \phi_{ij} & y_{ji} \\ \sin \phi_{ij} & -\cos \phi_{ij} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
Composite Operations

• Compounding and reversal operations can be combined to compute any sequence of relationships

• Recursive head-to-tail

\[ x_{il} = x_{ij} \oplus x_{jl} = x_{ij} \oplus (x_{jk} \oplus x_{kl}) = x_{ik} \oplus x_{kl} = (x_{ij} \oplus x_{jk}) \oplus x_{kl} \]

• Compounding operation is associative, but not commutative

• Combine compounding and reversal operations (head-to-head)

\[ x_{ij} \ominus x_{kj} = x_{ij} \oplus (\ominus x_{kj}) \]

• Tail-to-tail combinations come from observing two things from the same point: \[ x_{jk} = (\ominus x_{ij}) \oplus x_{ik} \]
Composite Operations

• To estimate the mean of a complex relationship, just solve the estimate equations recursively

• e.g., tail-to-tail

\[ \hat{x}_{jk} = \hat{x}_{ji} \oplus \hat{x}_{ik} = \Theta \hat{x}_{ij} \oplus \hat{x}_{ik} \]

\[ C(x_{jk}) \approx J \oplus \left[ \begin{array}{cc} C(x_{ji}) & C(x_{ji}, x_{ik}) \\ C(x_{ik}, x_{ji}) & C(x_{ik}) \end{array} \right] J^\top \approx J \oplus \left[ \begin{array}{cc} J \ominus C(x_{ij})J^\top \ominus & J \ominus C(x_{ij}, x_{ik}) \\ J \ominus C(x_{ik}, x_{ij})J^\top \ominus & J \ominus C(x_{ik}) \end{array} \right] J^\top \]
General spatial relationship

• For any spatial relationship among world locations

\[ y = g(x) \]

• The estimated mean and covariance of the relationship

\[ \hat{y} \approx g(\hat{x}) \]
\[ C(y) \approx G_x C(x) G^\top_x \]
Build/update the map

• The map changes when
  • An object (e.g., the robot) moves
  • New spatial information is obtained

• Assumption
  • New spatial information is obtained at discrete moments $k$ and is instantaneous
  • As an object moves, no measurements of external objects are made
Moving object

- The system dynamics model is given by

\[ x_k^{(-)} = f(x_{k-1}^{(+)}, y_{k-1}) \]

where

\[ y_{k-1} = u_{k-1} + w \]

\[ \hat{y}_{k-1} = u_{k-1} \]

\[ C(y_{k-1}) = C(w) \]
Moving object

- Given the estimates of the state vector and variance matrix at state $k-1$

$$\hat{x}_k(-) \approx f(\hat{x}_{k-1}^(+), \hat{y}_{k-1})$$

$$C(x_k(-)) \approx F_{(x,y)} \begin{bmatrix} C(x_{k-1}^(+)) & C(x_{k-1}^(+), y_{k-1}) \\ C(y_{k-1}, x_{k-1}^(+)) & C(y_{k-1}) \end{bmatrix} F_{(x,y)}^\top$$

$$F_{(x,y)} = [F_x \quad F_y] = \frac{\partial (f(x,y))}{\partial (x,y)} (\hat{x}_{k-1}^(+), \hat{y}_{k-1})$$
Moving object

- The robot makes an uncertain relative motion
  \[ x'_R = x_R \oplus y_R \]

- Thus, only a small portion of the map should be updated

\[
\hat{x}_{k-1}^{(+)} = \begin{bmatrix} \hat{x}_R \\ \hat{x}_R \end{bmatrix} \quad \hat{x}_k^{(-)} = \begin{bmatrix} \hat{x}'_R \\ \hat{x}'_R \end{bmatrix} \quad \hat{x}'_R \simeq \hat{x}_R \oplus \hat{y}_R
\]

\[
C(x_k^{(-)}) = \begin{bmatrix} \hat{x}'_R \mid C(x, x'_R) \\ C(x'_R, x) \mid C(x'_R) \end{bmatrix}
\]

\[
C(x'_R, x_i) \approx J_1 \oplus C(x_R, x_i)
\]
New spatial information (1)

- New object is added to the map

\[
\hat{x}^{(+)} = \begin{bmatrix} \hat{x}^{(-)} \\ \hat{x}_{n+1} \end{bmatrix} \\
C(x_k^{(-)}) = \begin{bmatrix}
C(x^{(-)}) & C(x, x_{n+1}) \\
C(x_{n+1}, x) & C(x_{n+1})
\end{bmatrix}
\]

If independent of the estimates of other object locations

\[
x_{n+1} = x_{\text{new}} \\
\hat{x}_{n+1} = \hat{x}_{\text{new}} \\
C(x_{n+1}) = C(x_{\text{new}}) \\
C(x_{n+1}, x_i) = C(x_{\text{new}}, x_i) = 0
\]

Otherwise

\[
x_{n+1} = g(x, z) \\
\hat{x}_{n+1} = g(\hat{x}, \hat{z}) \\
C(x_{n+1}) = G_x C(x) G_x^T + G_z C(z) G_z \\
C(x_{n+1}, x_i) = C(x_{n+1}, x_i) \\
C(x_{n+1}, x) = G_x C(x)
\]
New spatial information (2)

• An already-existing object is sensed, thus some constraints are added to the existing relationships
• The measurement

\[ z = h(x) + v. \]

• The expected value of the sensor value and its covariance

\[ \hat{z} \approx h(\hat{x}). \]

\[ C(z) \approx H_x C(x) H_x^T + C(v), \]

where

\[ H_x \triangleq \frac{\partial h_k(x)}{\partial x} \left( \hat{x}_k^{(-)} \right) \]
For example, if the sensor measures the relative location of the observed object

\[ z = x_{21} = \Theta x_2 \oplus x_1. \]

\[ \hat{z} = \hat{x}_{21} = \Theta \hat{x}_2 \oplus \hat{x}_1. \]

\[
C(z) = \Theta J \Theta \left[
\begin{array}{cc}
C(x_2) & C(x_2, x_1) \\
C(x_1, x_2) & C(x_1)
\end{array}
\right] \Theta J^T + C(v)
\]

Given the sensor model, the Kalman filter equations can be used for updating the state estimate

\[
\begin{align*}
\hat{x}_k^{(+)} &= \hat{x}_k^{(-)} + K_k \left[ z_k - h_k(\hat{x}_k^{(-)}) \right], \\
C(\hat{x}_k^{(+)}) &= C(\hat{x}_k^{(-)}) - K_k H_x C(\hat{x}_k^{(-)}), \\
K_k &= C(\hat{x}_k^{(-)}) H_x^T \left[ H_x C(\hat{x}_k^{(-)}) H_x^T + C(v)_k \right]^{-1}.
\end{align*}
\]
Example

a) The robot starts from [0,0,0] coinciding the world reference frame origin
b) The robot senses object #1.
c) The robot moves.
d) The robot senses a different object #2.
e) Now the robot senses object #1 again.
Step a)

• The stochastic map is initialized

\[ \hat{x} = [\hat{x}_R] = [0] \]

\[ \mathbf{C}(x) = [\mathbf{C}(x_R)] = [0] \]
Step b)

- Object #1 is sensed and added to the stochastic map

\[ \hat{x} = \begin{bmatrix} \hat{x}_R \\ \hat{x}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{z}_1 \end{bmatrix} \]

\[ C(x) = \begin{bmatrix} C(x_R) & C(x_R, x_1) \\ C(x_1, x_R) & C(x_1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & C(z_1) \end{bmatrix} \]
Step c)

• The robot moves and so the entry related to the robot is updated

\[
\hat{x} = \begin{bmatrix} \hat{x}_R \\ \hat{x}_1 \end{bmatrix} = \begin{bmatrix} \hat{y}_R \\ \hat{z}_1 \end{bmatrix}
\]

\[
C(x) = \begin{bmatrix} C(x_R) & C(x_R, x_1) \\ C(x_1, x_R) & C(x_1) \end{bmatrix} = \begin{bmatrix} C(y_R) & 0 \\ 0 & C(z_1) \end{bmatrix}
\]
Object #1 wrt robot frame

\[ \text{OBJ1}_R = (\ominus \text{ROBOT}_W) \oplus \text{OBJ1}_W \]
\[ = \text{WORLD}_R \oplus \text{OBJ1}_W \]
Step d)

- A new object is sensed, from the robot reference frame.
- The stochastic map is updated accordingly

\[ \hat{x} = \begin{bmatrix} \hat{x}_R \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} \hat{y}_R \\ \hat{z}_1 \\ \hat{y}_R \oplus \hat{z}_2 \end{bmatrix} \]

\[ C(x) = \begin{bmatrix} C(x_R) & C(x_R, x_1) & C(x_R, x_2) \\ C(x_1, x_R) & C(x_1) & C(x_1, x_2) \\ C(x_2, x_R) & C(x_2, x_1) & C(x_2) \end{bmatrix} = \begin{bmatrix} C(y_R) & 0 & C(y_R)J_{1\oplus}^T \\ 0 & C(z_1) & 0 \\ J_{1\oplus}C(y_R) & 0 & C(x_2) \end{bmatrix} \]
Sensing Object#2

- $\text{OBJ2}_W = \text{ROBOT}_W \oplus \text{OBJ2}_R$
Step e)

- \( \text{OBJ1}_W = \text{ROBOT}_W \oplus \text{OBJ1}_R \)
Combining observations (update)

- $OJB_{1w} = OJB_{1w}(\text{new}) \otimes OBJ_{1w}(\text{old})$
- $OJB_{1r} = OJB_{1r}(\text{new}) \otimes OBJ_{1r}(\text{old})$
Combining observations (update)

- $\text{ROBOT}_W(\text{new}) = \text{OJB1}_W \oplus (\ominus \text{OJB1}_R)$
- $\text{ROBOT}_W = \text{ROBOT}_W(\text{new}) \odot \text{ROBOT}_W(\text{old})$
Discussion

• Data association?
• Partial observation?
• Non-unimodal Gaussians?
• Complexity?
• What happens if data association is wrong?
• Dynamic landmarks?
• Can be applied to certain decision-making problems
• …