CSCE 574 ROBOTICS

Coordinate Systems
• Position representation is:

\[
^A P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}
\]
Orientation Representations

- Describes the rotation of one coordinate system with respect to another
Rotation Matrix

• Write the unit vectors of $B$ in the coordinate system of $A$.

• Rotation Matrix:

\[
\begin{bmatrix}
\hat{X}_B & \hat{Y}_B & \hat{Z}_B
\end{bmatrix}^B_A = 
\begin{bmatrix}
\hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\
\hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\
\hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A
\end{bmatrix}
\]
Properties of Rotation Matrix

\[ \begin{align*}
  B_R & = A_R^T \\
  A_R^T B_R & = I_3 \\
  A_R & = B_R^{-1} = A_R^T
\end{align*} \]
Coordinate System Transformation

\[
M = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} & p_x \\
    r_{21} & r_{22} & r_{23} & p_y \\
    r_{31} & r_{32} & r_{33} & p_z \\
    0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
    R & T \\
    0_{3\times1} & 1
\end{bmatrix}
\]

where \( R \) is the rotation matrix and \( T \) is the translation vector.
Rotation Matrix

- The rotation matrix consists of 9 variables, but there are many constraints. The minimum number of variables needed to describe a rotation is three.
Rotation Matrix-Single Axis

\[ R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \]

\[ R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \]

\[ R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
Fixed Angles

- One simple method is to perform three rotations about the axis of the original coordinate frame:
  - X-Y-Z fixed angles

\[ \hat{b}A R(\theta, \phi, \psi) = R_z(\psi) R_y(\phi) R_x(\theta) \]

\[ = \begin{bmatrix}
  \cos(\psi) \cos(\phi) & \cos(\psi) \sin(\phi) \sin(\theta) - \sin(\psi) \cos(\theta) & \cos(\psi) \sin(\phi) \cos(\theta) + \sin(\psi) \sin(\theta) \\
  \sin(\psi) \cos(\phi) & \sin(\psi) \sin(\phi) \sin(\theta) + \cos(\psi) \cos(\theta) & \sin(\psi) \sin(\phi) \cos(\theta) + \cos(\psi) \sin(\theta) \\
  -\sin(\phi) & \cos(\phi) \sin(\theta) & \cos(\theta) \cos(\psi)
\end{bmatrix} \]

- There are 12 different combinations
Inverse Problem

- From a Rotation matrix find the fixed angle rotations:

\[
\begin{pmatrix}
\cos(\psi)\cos(\phi) & \cos(\psi)\sin(\phi)\sin(\theta) - \sin(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\cos(\theta) + \sin(\psi)\sin(\theta) \\
\sin(\psi)\cos(\phi) & \sin(\psi)\sin(\phi)\sin(\theta) + \cos(\psi)\cos(\theta) & \sin(\psi)\sin(\phi)\cos(\theta) + \cos(\psi)\sin(\theta) \\
-\sin(\phi) & \cos(\phi)\sin(\theta) & \cos(\theta)\cos(\psi)
\end{pmatrix}
= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33} \end{pmatrix}
\]

thus:

\[
\phi = A \tan 2\left( - r_{31}, \sqrt{r_{11}^2 + r_{21}^2} \right)
\]

\[
\psi = A \tan 2\left( r_{21}/\cos(\phi), r_{11}/\cos(\phi) \right)
\]

\[
\theta = A \tan 2\left( r_{32}/\cos(\phi), r_{33}/\cos(\phi) \right)
\]
Euler Angles

• **ZYX**: Starting with the two frames aligned, first rotate about the $Z_B$ axis, then by the $Y_B$ axis and then by the $X_B$ axis. The results are the same as with using XYZ fixed angle rotation.

• There are 12 different combination of Euler Angle representations
Euler Angles

- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw
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Pitch
Euler Angles

- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw
Euler Angle concerns: Gimbal Lock

Using the ZYZ convention

• \((90°, 45°, -105°) \equiv (-270°, -315°, 255°)\)  multiples of 360°
• \((72°, 0°, 0°) \equiv (40°, 0°, 32°)\)  singular alignment (Gimbal lock)
• \((45°, 60°, -30°) \equiv (-135°, -60°, 150°)\)  bistable flip
Represent an arbitrary rotation as a combination of a vector and an angle.
Quaternions

- Are similar to axis-angle representation
- Two formulations
  - Classical
  - Based on JPL’s standards

- Avoids Gimbal lock

# Quaternions

## Classic notation

\[
\bar{q} = q_4 + q_1 i + q_2 j + q_3 k
\]

- \(i^2 = j^2 = k^2 = ijk = -1\)
- \(ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j\)

\[
\bar{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad q_0 = \cos\left(\frac{\theta}{2}\right), \quad q = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) \cos(\beta_x) \\ \sin\left(\frac{\theta}{2}\right) \cos(\beta_y) \\ \sin\left(\frac{\theta}{2}\right) \cos(\beta_z) \end{bmatrix}
\]

\[
\bar{q} \otimes \bar{p}, \quad q \times p, \quad \bar{q}_I, \quad [q \times]
\]

## JPL-based

\[
\bar{q} = q_4 + q_1 i + q_2 j + q_3 k
\]

- \(i^2 = j^2 = k^2 = -1\)
- \(-ij = ji = k, -jk = kj = i, -ki = ik = j\)

\[
\bar{q} = \begin{bmatrix} q_4 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad q = \begin{bmatrix} k_x \sin\left(\frac{\theta}{2}\right) \\ k_y \sin\left(\frac{\theta}{2}\right) \\ k_z \sin\left(\frac{\theta}{2}\right) \end{bmatrix}, \quad q_4 = \cos\left(\frac{\theta}{2}\right)
\]

\[
\|\bar{q}\| = 1, \quad \bar{q} \otimes \bar{p}, \quad q \times p, \quad \bar{q}_I, \quad [q \times]
\]

Coordinate frames on PR2