Today

Perspective Camera Models

Announcement

Homework 1 has been posted on Blackboard and the course website.

Due date: 1:15pm EST, Tuesday, Feb. 4

Late submission penalty applies!

You may need to read the related chapters in the book to do this homework

Announcement

Quiz #1 is available in Blackboard.

Due date: 11:59pm EST, Tuesday, Jan. 21

Open book and open notes

Basic Optics: Pinhole Cameras

Mount a piece of film in a lightproof box with a single pinhole in it

Pinhole focuses light on the film

- Lens degenerates to a point
- One-to-one correspondence between 3D object point and 2D image point
- Only select light ray can go through the hole (the hole is reduced to a point)
- Image on film is flipped upside down



Parallel Lines Meet at Vanishing Points

- <u>Horizon H</u> is the intersection of a plane that is parallel to Π and passes through the pin hole O
- <u>Vanishing point</u> is a point on the horizon H where the images of parallel lines on the plane Π intersects
- L has no image on the l₂ image plane

Object



Vanishing Points (cont.)

Each set of parallel lines (=direction) meets at a different point

 The vanishing point for this direction

Good ways to spot faked images

- scale and perspective don't work
- vanishing points behave badly

Sets of parallel lines on the same plane lead to *collinear* vanishing points.

 The line is called the horizon for that plane



Points, Lines, and Planes

Point: represented as a vector $\mathbf{p} = [p_1, p_2, \cdots, p_N]$ in Euclidean space \mathbb{R}^N

Line:

- Lines in 2D
 - Slope-intercept form: y = ax + b
 - $\mathbf{l} \cdot \mathbf{p} = 0$, where $\mathbf{l} = [a, b, c]$, and $\mathbf{p} = [x, y, 1]$

Lines in general

- Parametric form: $\mathbf{p}(t) = \mathbf{p}_0 + t \mathbf{d}$, where \mathbf{p}_0 is any point on the line and $\mathbf{d} = [d_1, d_2, \dots, d_N]$ is a unit vector - the direction of the line
 - Parallel lines have the same d with different \mathbf{p}_0
- Point-normal: $\mathbf{m} \cdot (\mathbf{p} \mathbf{p}_0) = 0$, where \mathbf{p}_0 is any point on the line and \mathbf{m} is normal to the line
- $\mathbf{l} \cdot \mathbf{p} = 0$, where $\mathbf{l} = [\mathbf{m}, -\mathbf{m} \cdot \mathbf{p}_0]$, and $\mathbf{p} = [p_1, p_2, \cdots, p_N, 1]$

Homogenous Coordinates

The line function: $l \cdot p = 0$, where l=[a, b, c] and p=[x, y, 1]

$$\implies 1 \cdot (\mathbf{k}\mathbf{p}) = 0$$

for 2D point (x,y)

• (x,y,1) is the same as (kx,ky,k)

for 3D point (x,y,z)

• (x,y,z,1) is the same as (kx,ky,kz,k)

Add an extra coordinate and has an equivalence relation

Homogenous Coordinates

Why we use homogenous coordinates?

- Possible to represent points "at infinity"
 - Where parallel lines intersect $-\mathbf{p}_{\infty} = (\mathbf{x}, \mathbf{y}, \mathbf{0})$ Point at infinity
 - Where parallel planes intersect $-l_{\infty} = (0,0,1)$ Line at infinity
- Possible to write the perspective projection as a matrix multiplication

Points, Lines, and Planes

Intersection point of two lines

$$\mathbf{p} = \mathbf{l_1} \times \mathbf{l_2} \qquad \text{How to prove?}$$

or p is in the null space of $\begin{bmatrix} \mathbf{l_1^T} \\ \mathbf{l_2^T} \end{bmatrix}$

Given two points, the line parameter vector can be found as

 $l = p_1 \times p_2$

Points, Lines, and Planes

Plane:

$$\mathbf{n}\cdot(\mathbf{p}-\mathbf{p}_0)=\mathbf{0}$$

n is the normal vector – perpendicular to the plane

 \mathbf{p}_0 is any point on the plane

From 3D to 2D: the Equation of Projection



Only one coordinate system – camera coordinate system

Terminology

Center of projection (O): the pinhole center

Focal length (f): the distance from the image plane to O

- (the distance between the lens center to the CCD array in practice)
- **Optical axis:** the line passing through O and perpendicular to the image plane
- Principal point: the intersection of the optical axis with the image plane

The Equation of Perspective Projection

Cartesian coordinates:

• We have, by similar triangles, that

$$P = (x, y, z) \rightarrow P' = \left(-f\frac{x}{z}, -f\frac{y}{z}, f\right)$$

 Ignoring the third coordinate and assuming the image plane is between the object and the projection center, we get

x f y Isotropic scaling

$$(x, y, z) \rightarrow (u, v) = (\frac{f}{z}x, \frac{f}{z}y)$$

scaling

3D object point \rightarrow 2D image point

The perspective projection is non-linear!

Perspective Projection Invariants



Projection of a Line

Lines project to lines

$$p(t) = p_0 + td$$
A 3D line
$$p_0 = [x_0, y_0, z_0]$$

$$d = [d_x, d_y, d_z]$$

$$[u(t) v(t)] = \left[\frac{f(x_0 + td_x)}{z_0 + td_z}, \frac{f(y_0 + td_y)}{z_0 + td_z}\right]$$
Is it a line?

Yes. But how to prove?

Projection of a Line

Lines project to lines

$$p(t) = p_{0} + td$$
A 3D line $p_{0} = [x_{0}, y_{0}, z_{0}]$
 $d = [d_{x}, d_{y}, d_{z}]$

$$[u(t) v(t)] = \left[\frac{f(x_{0} + td_{x})}{z_{0} + td_{z}}, \frac{f(y_{0} + td_{y})}{z_{0} + td_{z}}\right]$$

When the 3D line goes to infinity, $t \rightarrow \infty$

 $[u(t) v(t)] = \left[\frac{fd_x}{d_z}, \frac{fd_y}{d_z}\right] \implies \text{The vanishing point}$ Why parallel lines have the same vanishing point?

Proof: Projection of a 3D Line is Still a Line

Two cases:

Case1 : The line passes through a point (p_0) on the XY plane $\rightarrow z_0 = 0$

$$\mathbf{p}' = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \frac{f(x_0 + tdx)}{z_0 + tdz} \\ \frac{f(y_0 + tdy)}{z_0 + tdz} \end{bmatrix} = \begin{bmatrix} \frac{f(x_0 + tdx)}{tdz} \\ \frac{f(y_0 + tdy)}{tdz} \end{bmatrix} = \begin{bmatrix} \frac{fdx}{dz} + \frac{fx_0}{dz} \frac{1}{t} \\ \frac{fdy}{dz} + \frac{fy_0}{dz} \frac{1}{t} \end{bmatrix}$$
$$= \mathbf{p}'_0 + t'\mathbf{d}'$$
$$\mathbf{p}'_0 = \begin{bmatrix} \frac{fdx}{dz} & \frac{fdy}{dz} \end{bmatrix}^T \mathbf{t}' = \frac{f}{tdz'} \mathbf{d}' = \begin{bmatrix} x_0 & y_0 \end{bmatrix}^T \qquad \Rightarrow \begin{array}{l} \mathbf{p}' = \mathbf{p}'_0 + t'\mathbf{d}' \\ \mathbf{A} \text{ line} \end{array}$$
Vanishing point Foreshortening

Vanishing point

Proof: Projection of a 3D Line is Still a Line

Case2. The line is parallel to the XY plane $\rightarrow dz = 0$

$$\mathbf{p}' = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \frac{f(x_0 + tdx)}{z_0 + tdz} \\ \frac{f(y_0 + tdy)}{z_0 + tdz} \end{bmatrix} = \begin{bmatrix} \frac{f(x_0 + tdx)}{z_0} \\ \frac{f(y_0 + tdy)}{z_0} \end{bmatrix} = \mathbf{p}'_0 + t'd'$$
$$\mathbf{p}'_0 = \frac{f}{z_0} \begin{bmatrix} x_0 & y_0 \end{bmatrix}^T, \quad t' = \frac{f}{z_0} t, \quad d' = \begin{bmatrix} d_x & d_y \end{bmatrix}^T \implies A \text{ line}$$
Scaling

Properties of Perspective Projection

- Points project to points
- Lines project to lines
- **Cross-ratio is preserved after projection**
- Planes project to the whole or half image
 - A plane may only has half of its area in the projection side
- **Scaling and foreshortening**

Angles are not preserved

 Parallel lines may be not projected to parallel lines unless they are parallel to the image plane

Degenerate cases

- Line through center of projection projects to a point.
- Plane through center of projection projects to line





Perspective Projection using Matrix-Vector Form

Turn previous expression into homogeneous coordinates

- 3D point in homogeneous coordinates --- (X, Y, Z, 1)
- Projected 2D point in image in homogeneous coordinates --- $(\lambda x, \lambda y, \lambda)$

$$\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ f \end{bmatrix}$$

 $\lambda = Z/f$

Weak Perspective Projection

If the relative depth Δz between two 3D points P and Q is relative small (e.g., $\Delta z < \frac{\bar{z}}{20}$), $z \approx \bar{z}$

Usage:

- Perfect for planar object parallel to the image plane
- collect points into a group at about the same depth, then divide each point by the depth of its group
- Advantage: easy and is acceptable in many conditions

