Today

Perspective Camera Models

Announcement

Quiz #1 is available in Blackboard.

Due date: 11:59pm EST, Wednesday, Jan. 18th

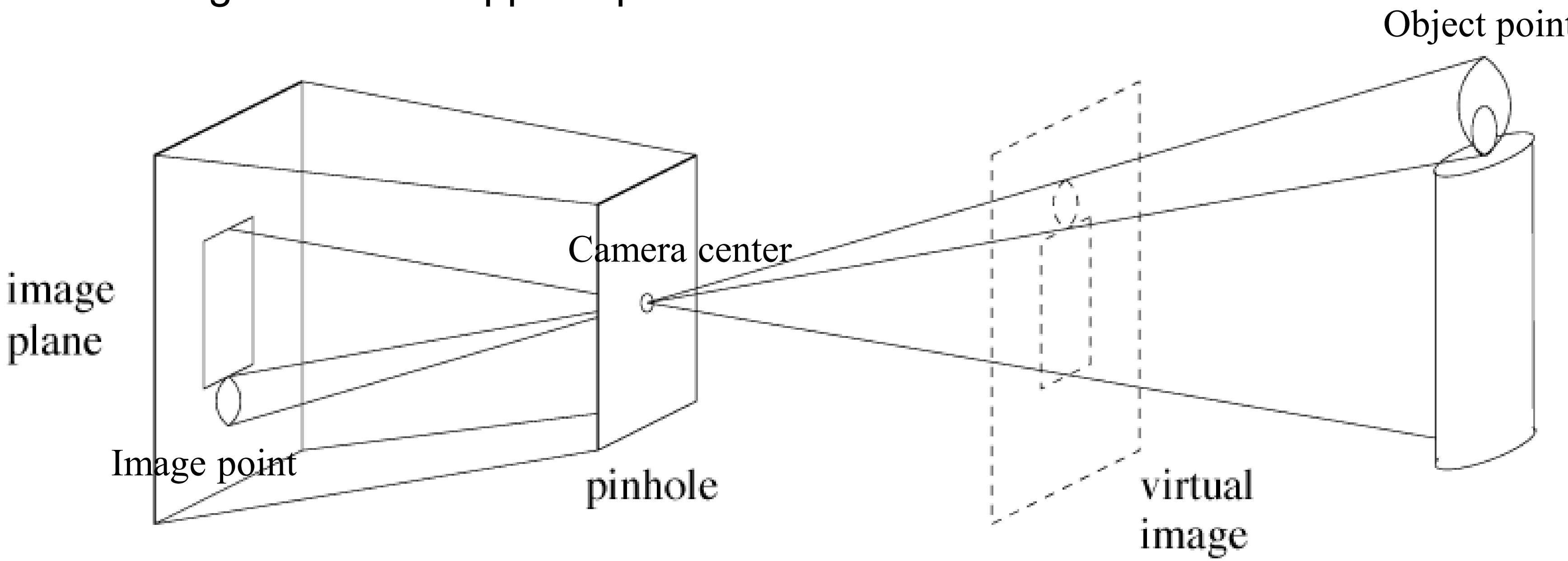
Open book and open notes

Basic Optics: Pinhole Cameras

Mount a piece of film in a lightproof box with a single pinhole in it

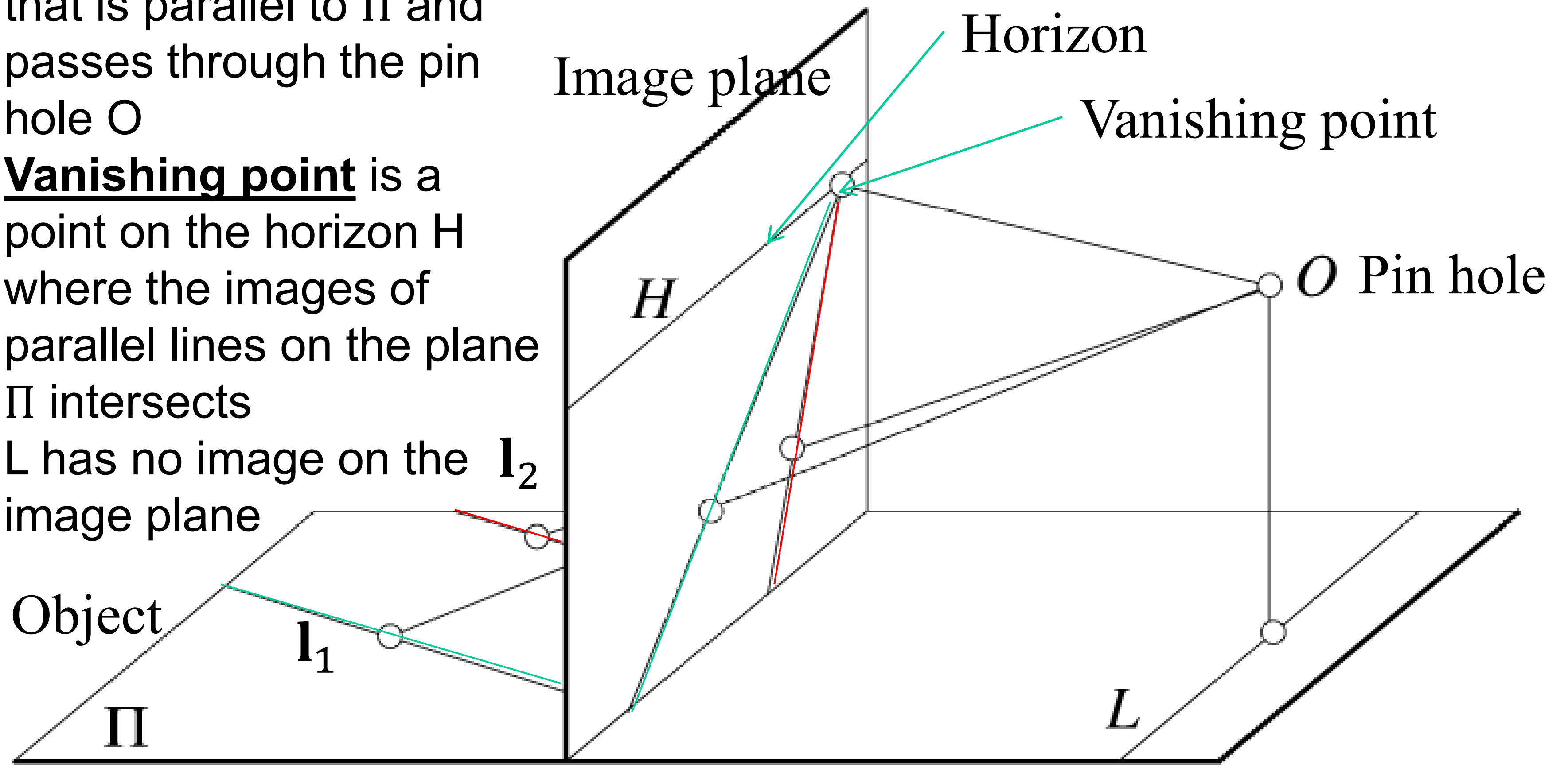
Pinhole focuses light on the film

- Lens degenerates to a point
- One-to-one correspondence between 3D object point and 2D image point
- Only select light ray can go through the hole (the hole is reduced to a point)
- Image on film is flipped upside down



Parallel Lines Meet at Vanishing Points

- **Horizon H** is the intersection of a plane that is parallel to Π and passes through the pin hole O
- **Vanishing point** is a point on the horizon H where the images of parallel lines on the plane Π intersects
- L has no image on the image plane



Vanishing Points (cont.)

Each set of parallel lines (=direction) meets at a different point

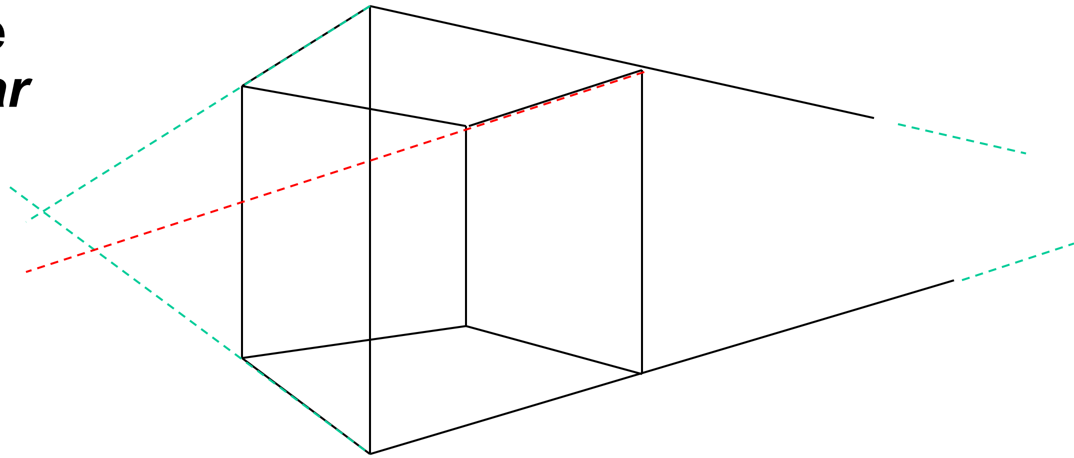
- The *vanishing point* for this direction

Sets of parallel lines on the same plane lead to *collinear* vanishing points.

- The line is called the *horizon* for that plane

Good ways to spot faked images

- scale and perspective don't work
- vanishing points behave badly



Points, Lines, and Planes

Point: represented as a vector $\mathbf{p} = [p_1, p_2, \cdots, p_N]$ in Euclidean space $\mathbb{R}^{\boldsymbol{N}}$

Line:

- **Lines in 2D**
  - Slope-intercept form: $y = ax + b$
  - $\mathbf{l} \cdot \mathbf{p} = 0$, where $\mathbf{l}$=$[a, b, c]$, and $\mathbf{p}$=$[x, y, 1]$
- **Lines in general**
  - Parametric form: $\mathbf{p}(t) = \mathbf{p}_0 + t\,\mathbf{d}$, where $\mathbf{p}_0$ is any point on the line and $\mathbf{d}$=$[d_1, d_2, \cdots, d_N]$ is a unit vector - the direction of the line
    - Parallel lines have the same $\mathbf{d}$ with different $\mathbf{p}_0$
  - Point-normal: $\mathbf{m} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$, where $\mathbf{p}_0$ is any point on the line and $\mathbf{m}$ is normal to the line
  - $\mathbf{l} \cdot \mathbf{p} = 0$, where $\mathbf{l}$=$[\mathbf{m}, -\mathbf{m} \cdot \mathbf{p}_0]$, and $\mathbf{p}$=$[p_1, p_2, \cdots, p_N, 1]$

Homogenous Coordinates

The line function: $\mathrm{l} \cdot \mathbf{p} = 0$, where $\mathrm{l}=[a, b, c]$ and $\mathbf{p}=[x, y, 1]$

⇨ $$\mathrm{l} \cdot (\boldsymbol{k}\mathrm{p}) = 0$$

for 2D point (x,y)

- (x,y,1) is the same as (kx,ky,k)

for 3D point (x,y,z)

- (x,y,z,1) is the same as (kx,ky,kz,k)

⇨ **Add an extra coordinate and has an equivalence relation**

Homogenous Coordinates

Why we use homogenous coordinates?

- Possible to represent points "at infinity"
  - Where parallel lines intersect – $\mathbf{p}_\infty = (\boldsymbol{x}, \boldsymbol{y}, \mathbf{0})$ Point at infinity
  - Where parallel planes intersect – $\mathrm{l}_\infty = (0,0,1)$ Line at infinity
- Possible to write the perspective projection as a matrix multiplication

Points, Lines, and Planes

Intersection point of two lines

$$\mathrm{p} = \mathrm{l}_1 \times \mathrm{l}_2 \qquad \text{How to prove?}$$

or p **is in the null space of** $\begin{bmatrix} \mathrm{l}_1^T \\ \mathrm{l}_2^T \end{bmatrix}$

Given two points, the line parameter vector can be found as

$$\mathrm{l} = \mathrm{p}_1 \times \mathrm{p}_2$$

Points, Lines, and Planes

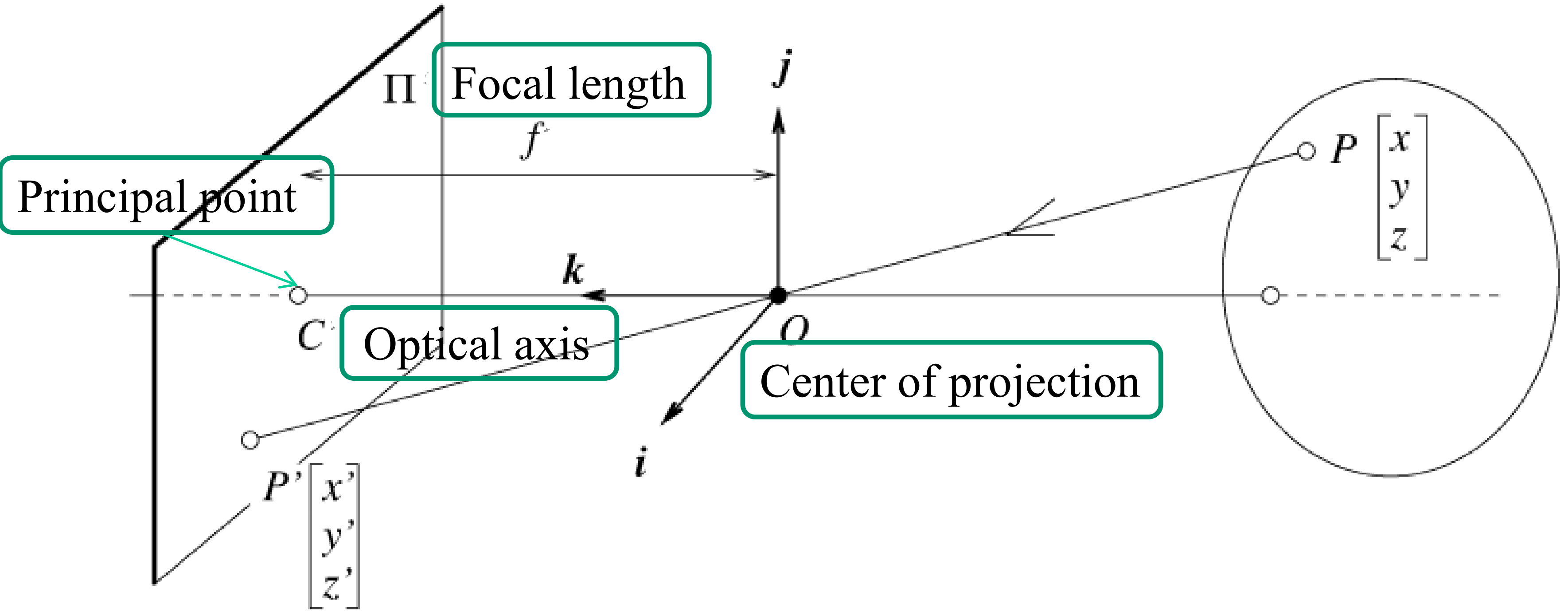
Plane:

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p_0}) = \mathbf{0}$$

$\mathbf{n}$ is the normal vector – perpendicular to the plane

p$_0$ is any point on the plane

From 3D to 2D: the Equation of Projection



Only one coordinate system – camera coordinate system

Terminology

Center of projection (O): the pinhole center

Focal length (f): the distance from the image plane to O

- (the distance between the lens center to the CCD array in practice)

Optical axis: the line passing through O and perpendicular to the image plane

Principal point: the intersection of the optical axis with the image plane

The Equation of Perspective Projection

Cartesian coordinates:

- We have, by similar triangles, that

$$P = (x, y, z) \rightarrow P' = (-f\frac{x}{z}, -f\frac{y}{z}, f)$$

- Ignoring the third coordinate and assuming the image plane is between the object and the projection center, we get

$$(x, y, z) \rightarrow (u, v) = (\frac{f}{z}x, \boxed{\frac{f}{z}}y)$$

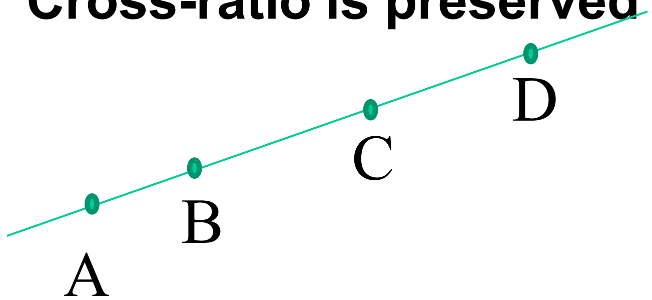
Isotropic scaling

3D object point → 2D image point

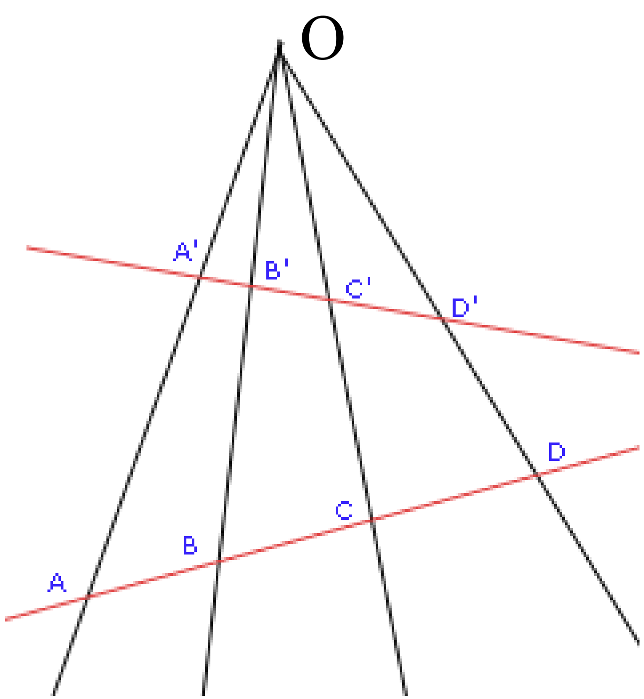
The perspective projection is non-linear!

Perspective Projection Invariants

Cross-ratio is preserved under perspective projection



$$(A,B;C,D) = \frac{AC}{BC} \Big/ \frac{AD}{BD}$$



$$(A,B;C,D) = (A',B';C',D')$$

How to prove it? A homework question.

Projection of a Line

Lines project to lines

$$\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{d}$$

A 3D line

$$\mathbf{p_0} = [x_0, y_0, z_0]$$

$$\mathbf{d} = [d_x, d_y, d_z]$$

$$\downarrow$$

$$[u(t)\ v(t)] = \left[\frac{f(x_0 + td_x)}{z_0 + td_z}, \frac{f(y_0 + td_y)}{z_0 + td_z}\right]$$

Is it a line?

Yes. But how to prove?

Projection of a Line

Lines project to lines

A 3D line
$$\mathbf{p}(\mathrm{t}) = \mathbf{p_0} + t\mathbf{d}$$
$$\mathbf{p_0} = [x_0, y_0, z_0]$$
$$\mathbf{d} \;\; = [d_x, d_y, d_z]$$

$$[u(t)\ v(t)] = \left[\frac{f(x_0 + td_x)}{z_0 + td_z}, \frac{f(y_0 + td_y)}{z_0 + td_z}\right]$$

When the 3D line goes to infinity, $t \to \infty$

$$[u(t)\ v(t)] = \left[\frac{fd_x}{d_z}, \frac{fd_y}{d_z}\right]$$

The vanishing point

Why parallel lines have the same vanishing point?

Proof: Projection of a 3D Line is Still a Line

Two cases:

Case1 : The line passes through a point $(\mathbf{p_0})$ on the XY plane → $z_0 = 0$

$$\mathbf{p}' = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \frac{f(x_0 + tdx)}{z_0 + tdz} \\ \frac{f(y_0 + tdy)}{z_0 + tdz} \end{bmatrix} = \begin{bmatrix} \frac{f(x_0 + tdx)}{tdz} \\ \frac{f(y_0 + tdy)}{tdz} \end{bmatrix} = \begin{bmatrix} \frac{fdx}{dz} + \frac{fx_0}{dz}\frac{1}{t} \\ \frac{fdy}{dz} + \frac{fy_0}{dz}\frac{1}{t} \end{bmatrix}$$

$$= \mathbf{p}_0' + t'\mathbf{d}'$$

$$\mathbf{p}_0' = \begin{bmatrix} \frac{fdx}{dz} & \frac{fdy}{dz} \end{bmatrix}^T, t' = \frac{f}{tdz}, \mathbf{d}' = \begin{bmatrix} x_0 & y_0 \end{bmatrix}^T$$

Vanishing point

→ $\mathbf{p}' = \mathbf{p}_0' + t'\mathbf{d}'$

A line

Foreshortening

Proof: Projection of a 3D Line is Still a Line

Case2. The line is parallel to the XY plane → $\boldsymbol{dz = 0}$

$$\mathbf{p}' = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \dfrac{f(x_0 + tdx)}{z_0 + tdz} \\ \dfrac{f(y_0 + tdy)}{z_0 + tdz} \end{bmatrix} = \begin{bmatrix} \dfrac{f(x_0 + tdx)}{z_0} \\ \dfrac{f(y_0 + tdy)}{z_0} \end{bmatrix} = \mathbf{p}_0' + t'\boldsymbol{d}'$$

$$\mathbf{p}_0' = \frac{f}{z_0}[x_0 \quad y_0]^T, \; t' = \frac{f}{z_0} t, \; \mathbf{d}' = [d_x \quad d_y]^T$$ ⇒ A line

Scaling

Properties of Perspective Projection

Points project to points

Lines project to lines

Cross-ratio is preserved after projection

Planes project to the whole or half image

- A plane may only has half of its area in the projection side

Scaling and foreshortening

Angles are not preserved

- Parallel lines may be not projected to parallel lines unless they are parallel to the image plane

Degenerate cases

- Line through center of projection projects to a point.
- Plane through center of projection projects to line