Addition by QFT.

* QFT. \( \text{QFT}_n : 1 \rightarrow \frac{1}{\sqrt{n}} \sum_{k=0}^{2^n-1} e^{2\pi i \frac{a \cdot k}{2^n}} |k\rangle \)

\[ a = a_n 2^{n-1} + a_{n-1} 2^{n-2} + \cdots + a_0 2^0. \]

Remarks: 1. \( \text{QFT}_n \) is unitary (linear & reversible).

2. \( \text{QFT}_n(a) = \phi_n(a) \otimes \phi_{n-1}(a) \cdots \otimes \phi_1(a) \)

where  \( \phi_k(a) = \frac{1}{\sqrt{n^2}} \left( |0\rangle + e^{2\pi i \frac{a}{2^k}} |1\rangle \right) \)

Note that we write \( \exp(x) = e^{2\pi i x} \) and

\[ \frac{a}{2^k} = a_n a_{n-1} \cdots a_{k+1} \cdot a_k a_{k-1} \cdots a_1, \text{ and} \]

\[ \exp(a_n a_{n-1} \cdots a_{k+1} \cdot a_k a_{k-1} \cdots a_1) = \exp(a \cdot a_n a_{n-1} \cdots a_1) \]

because \( \exp(l) = 1 \) if \( l \) is an integer.

So if we can construct a circuit for \( \phi_k \), then we have a circuit for \( \text{QFT}_n \).
\[ H |x\rangle = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle) \]
\[ = \frac{1}{\sqrt{2}} (|0\rangle + \exp(\frac{x}{2}) |1\rangle) \]
\[ = \frac{1}{\sqrt{2}} (|0\rangle + \exp(0 \cdot x) |1\rangle) \]

Let us define \( R_r (|x\rangle) = e^{\frac{2 \pi i}{2^r} x} |x\rangle \).

So, \( R_r = \exp\left(0 \cdot 0 \cdots \frac{x}{2^r}\right) |x\rangle \).

And controlled-\( R_r \)

\[ CR_r |ab\rangle = \exp\left(\frac{ab}{2^r}\right) |ab\rangle \]

Let's consider \( \phi_n (a) = \frac{1}{\sqrt{2}} (|0\rangle + \exp(0.a_n \cdots a_0) |1\rangle) \)
\[ = \frac{1}{\sqrt{2}} (|0\rangle + \exp(0.a_n + 0.a_{n-1} + \cdots + 0 \cdot 0 \cdots 0.a_0) |1\rangle) \]

So, the following implement \( \phi_n \).
So $QFT_n$ can be implemented by a circuit of the following form:

![Circuit Diagram]

Now imagine. Note that we have

\[ \phi_n(atb) \quad \phi_{n+1}(a+b) \]

So if we can figure out how to construct $\phi_k(atb)$, then we can apply $QFT_n^\dagger$ to obtain $atb$.

For example, $\phi_n(at+b)$ can be realized by the following circuit:

\[ \phi_n(atb) = |0\rangle + \exp\left(\frac{atb}{2^n}\right)|1\rangle \]
\[ = |0\rangle + \exp\left(\frac{a}{2^n} + \frac{b}{2^n}\right)|1\rangle \]
\[ = |0\rangle + \exp\left(0.a_n + \ldots + 0.0 \ldots 0a_1 + 0.b_n + \ldots + 0.0 \ldots b_1\right)|1\rangle \]
So, the general scheme of doing QFT-addition is the following.

* Remark: 1. We basically shift the problem of addition to phase, and we do phase addition by using CR_n gate. Then we use QFT to retrieve the result.

2. The QFT-addition scales nicely with multiple additions, e.g. a+b+c will just be adding another layer of CRs.