Extensions of Simply typed Lambda Calculus (STLC)

STLC is fairly limited. There are a few types and terms we can add to STLC to make it more expressive.

* Empty Type: \( \bot \)

\[ \Gamma \vdash M : \bot \]
\[ \Gamma \vdash \text{abort}_A M : A \]

by Curry-Howard correspondence, \( M \) is a proof of contradiction, so we can use \( \text{abort}_A M \) to cast \( M \) to any type \( A \).

\[ \text{abort}_A \bot \not\sim \text{error}. \]

* Unit Type: Unit.

\[ \Gamma \vdash () : \text{Unit} \]

Unit is like 'void' type in C.

By Curry-Howard, Unit type corresponds to True, or \( \top \), which only have a unique value, ()
* Sum Type. \( A + B \).

\[ \Gamma \vdash M : A \quad \Gamma \vdash M : B \]

\[ \Gamma \vdash \text{left } M : A + B \]

\[ \Gamma \vdash \text{right } M : A + B \]

\[ \Gamma \vdash M : A + B \quad \Gamma, x : A \vdash N_1 : C \quad \Gamma, y : B \vdash N_2 : C \]

\[ \Gamma \vdash \text{case } M \text{ of } \]

\[ \text{left } x \rightarrow N_1 \]

\[ \text{right } y \rightarrow N_2 \]

called "case expression".

under Curry-Howard, \( A + B \) also mean disjunction, i.e. \( A \) or \( B \).

Values: \( V ::= \ldots \text{ left } V \mid \text{ right } V \).

Evaluation:

\[ (\text{case( } \text{left } V \text{ ) of} \]

\[ \begin{align*}
\text{left } x & \rightarrow N_1 \\
\text{right } y & \rightarrow N_2
\end{align*} \]

\[ \rightarrow [V/x] \text{ N}_1 \]

similarly for (case( right V ) of )
So: $\text{STLC with Unit, } \bot, A+B$ correspond

Note that $\text{Bool}$ can actually be defined as

$$\text{Bool} := \text{Unit} + \text{Unit}.$$  
$$\text{true} := \text{left}()$$  
$$\text{false} := \text{right}().$$

$$\text{if } M \text{ then } N_1 \text{ else } N_2 :=$$

$$\text{case } M \text{ of }$$  
$$\text{left } x \rightarrow N_1$$  
$$\text{right } y \rightarrow N_2.$$  

@Note: $\text{STLC with Unit, } \bot, A+B$ corresponds to "intuitionistic propositional logic".

The main difference

$\forall \ast. \text{ In intuitionistic logic, we define } \neg A := A \to \bot.$
* In intuitionistic logic, we don't have

\[(A \rightarrow B) \Leftrightarrow (\neg A + B), \text{ i.e.}\]

there is no term \( M \) s.t.

\[\vdash M : ((A \rightarrow B) \rightarrow (\neg A + B)) \times (\neg A + B) \rightarrow (A \rightarrow B)\]

* There are two other main differences with classical logic.

1. in intuitionistic logic, we can't prove "law of excluded middle"

2. There is no term \( M \) s.t.

\[\vdash M : A + \neg A\]

2. There is no "The following is not a rule in intuitionistic logic"

\[\Gamma, x : \neg A \vdash M : \bot\]

\[\Gamma \vdash \text{contrad} : A\]

\[\Gamma \vdash \text{contrad} \cdot M\]