* Set and functions.

$a \in A$

$$f : A \to B$$

* Monoid.

$$(A,\ e : A,\ * : A \times A \to A)$$

① $\forall a \in A,\quad e * a = a * e = a.$

② $\forall a, b, c \in A.\quad (a * b) * c = a * (b * c)$

e.g. $(\mathbb{N},\ 0,\ +)$

$(\mathbb{N},\ 1,\ \times)$

$(\text{String},\ \text{""},\ \text{++})$

$(A \Rightarrow A,\ id_A : A \Rightarrow A,\ \circ)$

$f, g \in A \Rightarrow A$

$(f \circ g)(a) = f(g(a))$

* Monoid homomorphism.

$f : A \to B$ s.t.

$f(e_A) = e_B$

$\forall a, b \in A.\ f(a *_A b) = f(a) *_B f(b)$ $\left(\text{Note: } f(a), f(b) \in B\right)$

ex1. $f : (\mathbb{N}, 0, +) \to (\mathbb{N}, 1, *)$

$$f(0) = 1$$
$$f(a+b) = f(a) * f(b)$$

$$f(x) = 2^x$$

ex2: $\text{length} : [a] \to \mathbb{N}$

* Category. $\mathcal{C}$

① A collection of objects. $A \in obj(\mathcal{C})$

$A \in \mathcal{C}$

② $\forall A, B \in \mathcal{C}$, there is a set

$Hom(A, B)$.

$f \in Hom(A, B)$,

f is called a morphism from A to B.

denoted by $f : A \to B$

③ $\forall f : A \to B$, $g : B \to C$, $h : C \to D$

$\exists\, g \circ f : A \to C$ s.t.

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

④ $\forall A$, $id_A : A \to A$ s.t.

$$f \circ id_A = f \quad \forall f : A \to B.$$

and $$id_A \circ g = g \quad \forall g : C \to A$$

$$f : A \to B$$

domain of f codomain of f.

* Functors

A "functor" F from $\mathcal{C} \to \mathcal{D}$

denoted by $F : \mathcal{C} \to \mathcal{D}$

$\forall A \in \mathcal{C}$, $F(A) \in \mathcal{D}$

$\forall f : A \to B$ in $\mathcal{C}$, we have a

morphism $F(f) : F(A) \to F(B)$ s.t.

$$F(id_A) = id_{F(A)} : F(A) \to F(A)$$

$$F(f \circ g) = F(f) \circ F(g).$$

$$F_{AB} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

* $F : \mathcal{C} \to \mathcal{C}$, F is called end-functor

* Composition of functors.

$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$

$G \circ F : \mathcal{C} \to \mathcal{E}$

$$\begin{cases} \forall A \in \mathcal{C} & G \circ F(A) = G(F(A)) \in \mathcal{E} \\ \forall A, B \in \mathcal{C}, \ \forall f \in Hom_{\mathcal{C}}(A, B) & \\ \quad G \circ F(f) = G(\underbrace{F(f)}_{Hom_{\mathcal{D}}(F(A), F(B))}) \in Hom_{\mathcal{E}}(G(F(A)), G(F(B))). & \end{cases}$$

$G \circ F$ is indeed a functor.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$F : \mathcal{C} \to \mathcal{C}$

$G : \mathcal{C} \to \mathcal{C}$

$F \circ G$ or $G \circ F$

* Natural transformation.

Suppose $F, G : \mathcal{C} \to \mathcal{D}$

A natural transformation $\alpha : F \to G$ is a family of morphisms in $\mathcal{D}$.

denoted by $\boxed{\{\alpha_A : \underbrace{F(A)}_{\in \mathcal{D}} \to \underline{G(A)}\}_{A \in \mathcal{C}}}$,

such that

$\forall f : A \to B \in \mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd} \quad \forall \; f : A \to B.$$

in $\mathcal{D}$

in Haskell. α is like a polymorphic function $\alpha : F\,\underset{\sim}{a} \to G\,\underset{\sim}{a}$

$\star$ $F, G, H : \mathcal{C} \to \mathcal{D}$.

Suppose $(\alpha_A : F(A) \to G(A))_{A \in \mathcal{C}}$

$\alpha : F \to G$

$\beta : G \to H$

$(\beta_A : G(A) \to H(A))_{A \in \mathcal{C}}$

$\beta \circ \alpha : F \to H$

$= (\beta_A \circ \alpha_A : F(A) \to H(A))_{A \in \mathcal{C}}$

$\forall f : A \to B$.

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\beta_A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\
F(B) \arrow[r, "\alpha_B"] & G(B) \arrow[r, "\beta_B"] & H(B)
\end{tikzcd}$$

* Monad is "just" a monoid in the <u>category of endo-functors.</u>

Let $\mathcal{C}$ be a category.

We define a category $[\mathcal{C},\mathcal{C}]$ s.t.

obj($[\mathcal{C},\mathcal{C}]$) are endofunctors

$F : \mathcal{C} \to \mathcal{C}$

the morphisms are the natural transformations.

$$\begin{tikzcd}
F \arrow[loop left] \arrow[r] & G \arrow[r] & H
\end{tikzcd}$$

$Id : F \to F$

* Monad in $[\mathcal{C}, \mathcal{C}]$.

$$(M,\ \mu : M \circ M \to M,\ \eta : \mathrm{Id} \to M)$$

s.t. $\forall A \in \mathcal{C}$

$$\begin{tikzcd}
M(M(M(A))) \arrow[r, "\mu_{MA}"] \arrow[d, "M\mu_A"'] & M(M(A)) \arrow[d, "\mu_A"] \\
M(M(A)) \arrow[r, "\mu_A"'] & M(A)
\end{tikzcd}$$

$$\left(\mu_A : (M \circ M)(A) \to M(A) \right)_{A \in \mathcal{C}}$$

$$\begin{tikzcd}
M(A) \arrow[r, "\eta_{M(A)}"] \arrow[d, "M(\eta_A)"'] \arrow[dr, equal] & M(M(A)) \arrow[d, "\mu_A"] \\
M(M(A)) \arrow[r, "\mu_A"'] & M(A)
\end{tikzcd}$$

$$(\eta_A : A \to MA)_{A \in \mathcal{C}}$$

$$(Y,\ * : Y \times Y \to Y,\ e : 1 \to Y)$$