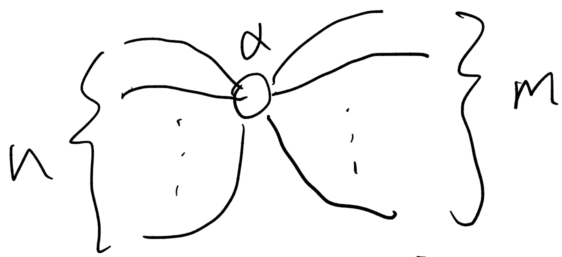
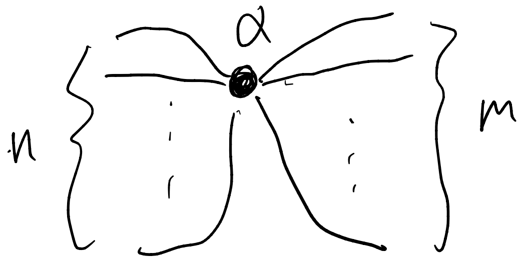
* ZX diagram (spiders + structural maps)
representing linear maps $Q^n \to Q^m$

* "spiders" (reading from left to right)



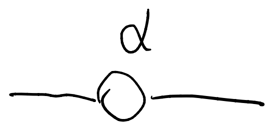
for $|\overbrace{0\cdots 0}^{m}\rangle\langle\overbrace{0\cdots 0}^{n}| + e^{i\alpha}|\overbrace{1\cdots 1}^{m}\rangle\langle\overbrace{1\cdots 1}^{n}|$

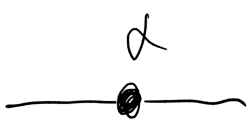
[$\alpha \in [0, 2\pi)$
we drop α when $\alpha = 0$]



for $|\overbrace{++\cdots+}^{m}\rangle\langle\overbrace{+\cdots+}^{n}| + e^{i\alpha}|-\cdots-\rangle\langle-\cdots-|$

$\frac{1}{\sqrt{2}}(|0\rangle + \cancel{|1\rangle} + |0\rangle - \cancel{|1\rangle})$

Note ①  : is the usual phase gate $P(\alpha)$ (Z-rotation gate)

②  is also unitary.
denoted by $R_X(\alpha)|0\rangle = |+\rangle\langle+|0\rangle + e^{i\alpha}|-\rangle\langle-|0\rangle$

$$= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|-\rangle = \frac{1}{2}\left(|0\rangle + |1\rangle + e^{i\alpha}|0\rangle - e^{i\alpha}|1\rangle\right)$$

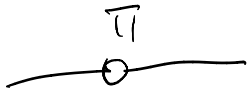
$$= \frac{1}{2}(1+e^{i\alpha})|0\rangle + \frac{1}{2}(1-e^{i\alpha})|1\rangle$$

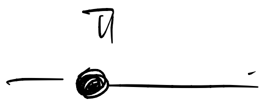
$$R_X(\alpha)|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}e^{i\alpha}|-\rangle$$

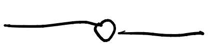
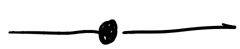
$$= \frac{1}{2}\left(|0\rangle + |1\rangle - e^{i\alpha}|0\rangle + e^{i\alpha}|1\rangle\right)$$

$$= \frac{1}{2}(1-e^{i\alpha})|0\rangle + \frac{1}{2}(1+e^{i\alpha})|1\rangle$$

X-rotation gate

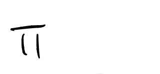
(3)  is Z-gate

 is X-gate

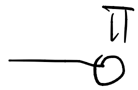
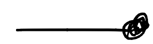
(4)  =  = 

is identity.

(5)  $\sqrt{2}|+\rangle$  $\sqrt{2}|0\rangle$

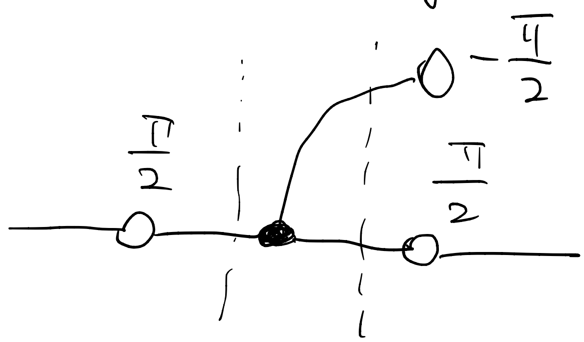
 $\sqrt{2}|-\rangle$  $\sqrt{2}|1\rangle$.

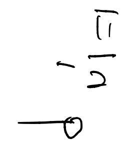
⑥  $\sqrt{2}\langle +|$  $\sqrt{2}\langle 0|$

 $\sqrt{2}\langle -|$  $\sqrt{2}\langle 1|$

⑦ We often ignore the nonzero scalar, unless we are calculating the amplitude of a state.

* Hadamard gate

H = 

 $\langle 0| + e^{-\frac{\pi}{2}i}\langle 1|$

$= \langle 0| - i\langle 1|$

$= P$

$S = \underline{|0\rangle\langle 0| + i|1\rangle\langle 1|}$; $|++\rangle\langle +| + |--\rangle\langle -|$; $P \otimes S$

$|0\rangle \xrightarrow{S} |0\rangle \longmapsto |++\rangle\langle +|0\rangle + |--\rangle\langle -|0\rangle$

$= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle \longmapsto \frac{1}{\sqrt{2}}(P|+\rangle \otimes S|+\rangle$

$+ P|-\rangle \otimes S|-\rangle)$

Note that

$$P|+\rangle = \frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)(|0\rangle + |1\rangle) = \frac{(1-i)}{\sqrt{2}}$$

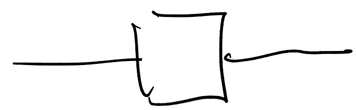
$$P|-\rangle = \frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)(|0\rangle - |1\rangle) = \frac{(1+i)}{\sqrt{2}}$$

So $\frac{1}{\sqrt{2}}(P|+\rangle \otimes S|+\rangle + P|-\rangle \otimes S|-\rangle)$

$$= \frac{1}{2\sqrt{2}}\left(\underline{(1-i)(|0\rangle + i|1\rangle)} + \underline{(1+i)(|0\rangle - i|1\rangle)} \right)$$

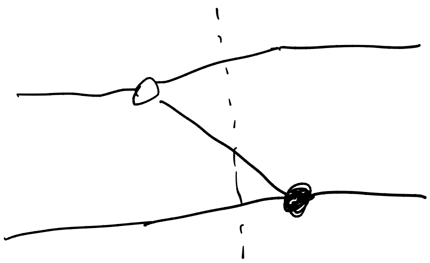
$$= \frac{1}{2\sqrt{2}}\left(2|0\rangle + ((i+1) - i + 1)\,|1\rangle \right)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

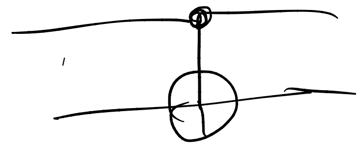
We often just write 

for Hadamard-gate in ZX-diagram.

* CNOT gate



for



why? $(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes I$;

$I \otimes (|+\rangle\langle ++| + |-\rangle\langle --|)$

$|00\rangle \mapsto |00\rangle \otimes |0\rangle = |0\rangle \otimes |00\rangle$

$\mapsto |0\rangle (|+\rangle\langle ++|00\rangle + |-\rangle\langle --|00\rangle)$

Note that $\langle ++| = \langle +| \otimes \langle +|$ $\quad \langle +|1\rangle = 1$

$\langle ++|00\rangle = \langle +|0\rangle \otimes \langle +|0\rangle$ $\quad \langle -|1\rangle = -1$

$|0\rangle + |1\rangle$

$+|0\rangle - |1\rangle$

$= 1/2$

$= \frac{1}{2}|0\rangle(|+\rangle + |-\rangle) = \frac{1}{\sqrt{2}\,2}(|0\rangle(2|0\rangle))$

$= \frac{1}{\sqrt{2}}|00\rangle$

$|01\rangle \mapsto |00\rangle \otimes |1\rangle = |0\rangle \otimes |01\rangle$

$\mapsto |0\rangle (|+\rangle\langle ++|01\rangle + |-\rangle\langle --|01\rangle)$

$= |0\rangle(\ (|+\rangle - |-\rangle)$

$= \frac{1}{\sqrt{2}} |01\rangle$.

$|0\rangle + |1\rangle - (|0\rangle - |1\rangle)$

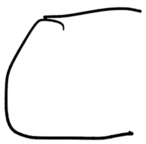
$= 2|1\rangle$

$|10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$

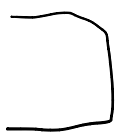
$|11\rangle \mapsto \frac{1}{\sqrt{2}} |10\rangle$ etc.

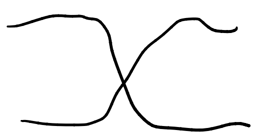
* So spiders with horizontal and vertical composition is enough to represent all unitary operations. (up-to-scalars).

* Structural maps.

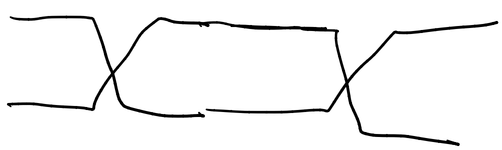
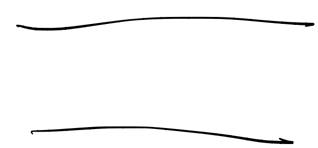
 for $|00\rangle + |11\rangle$ (bell state).

 for identity map.

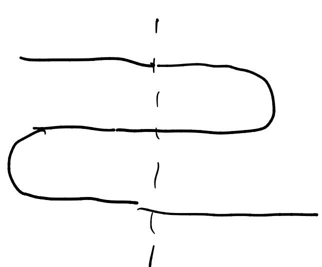
 for $\langle 00| + \langle 11|$ (bell meas)

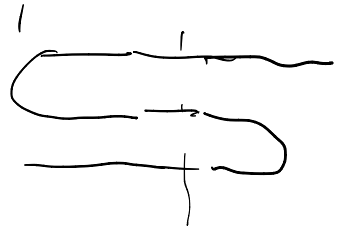
 for swap.

* thm:

 = 

* thm:

 = 

=  = 

$I \otimes (|00\rangle + |11\rangle) ; (\langle 00| + \langle 11|) \otimes I$.