(simple)
* Extensions of Simply typed Lambda Calculus (STLC):

STLC is fairly limited.
There are a few types and terms we can add to STLC to make it more expressive.

* Empty Type: $\perp$

$$\frac{\Gamma \vdash M : \perp}{\Gamma \vdash \text{abort}_A\, M : A}$$

by Curry-Howard correspondent, M is a proof of contradiction, so we can use $\text{abort}_A\, M$ to cast M to any type A.

$$\text{abort}_A\, V \rightsquigarrow \underline{\text{error}}.$$

* Unit Type Unit.

$$\frac{}{\Gamma \vdash () : \text{Unit}}$$

Unit is like 'void' type in C,
By Curry-Howard, Unit type corresponds to True, or $\top$, which only have a unique value, ().

* Sum Type. $A+B$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{left}\ M : A+B} \qquad \frac{\Gamma \vdash M : B}{\Gamma \vdash \mathrm{right}\ M : A+B}$$

$$\frac{\Gamma \vdash M : A+B \qquad \Gamma, x:A \vdash N_1 : C \qquad \Gamma, y:B \vdash N_2 : C}{\Gamma \vdash \begin{array}{l}\mathrm{case}\ M\ \mathrm{of} \\ \quad \mathrm{left}\ x \to N_1 \\ \quad \mathrm{right}\ y \to N_2\end{array}}$$

'x' is ~~bo~~ a bound variable, its scope is $N_1$

'y' is also bound, its scope is $N_2$

called "case expression".

under Curry-Howard, $A+B$ also mean disjunction, i.e. A or B.

Values : $V ::= \cdots \mid \mathrm{left}\ V \mid \mathrm{right}\ V$.

evaluation :

$$\frac{}{\left(\begin{array}{l}\mathrm{case}(\mathrm{left}\ V)\ \mathrm{of} \\ \quad \mathrm{left}\ x \to N_1 \\ \quad \mathrm{right}\ y \to N_2\end{array}\right) \rightsquigarrow [V/x]N_1}$$

similarly for $\left(\mathrm{case}(\mathrm{right}\ V)\ \mathrm{of}\ \cdots\right)$

~~So: STLC with Unit, ⊥, A+B correspon~~

Note that Bool can actually be defined as

$$\text{Bool} := \text{Unit} + \text{Unit}.$$

$$\text{true} := \text{left}()$$

$$\text{false} := \text{right}().$$

if M then $N_1$ else $N_2$ :=

case M of

  left $x \to N_1$

  right $y \to N_2$.

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Note: STLC with Unit, ⊥, A+B corresponds to "intuitionistic propositional logic".

~~The~~ ~~The main difference~~

*. In intuitionistic logic, we define $\neg A := A \to \bot$.

* In intuitionistic logic, we don't have

$$(A \to B) \Leftrightarrow (\neg A + B), \text{ i.e.}$$

there is no term $M$ s.t.

$$\vdash M : ((A \to B) \to (\neg A + B)) \times ((\neg A + B) \to (A \to B))$$

*. There are two other main differences with classical logic.

1. in intuistionistic logic, we can't prove "law of excluded middle".

There is no term $M$ s.t.

$$\vdash M : A + \neg A.$$

2. ~~There is no~~ "The following is not a rule in intuitionistic logic.

$$\frac{\Gamma, x : \neg A \vdash M : \bot}{\Gamma \vdash \cancel{\text{contra}\,M\,(x)} : A}$$

$\text{contra}\, x.\, M$