Programming Language Theory

* Simply typed lambda calculus.

$$\Gamma \vdash M : A$$

The term/program M has type A under the environment Γ.

What is a type?

$$A ::= \iota \mid A \to B \mid A \times B \mid \mathrm{Unit}$$

where ι denotes a built-in type: ~~set~~ e.g. Boolean, Integer.

What is a typing environment?

Γ is a list of distinct variables and their types

e.g. $[x_1 : A_1, \ldots x_n : A_n]$

What is a term/Program?

$$M ::= x \mid \underbrace{\lambda x.M}_{\text{lambda abstraction}} \mid \underbrace{M M'}_{\text{function application}} \mid \underbrace{(M, M')}_{\text{pair}} \mid \mathrm{fst}\, M \mid \mathrm{snd}\, M \mid ()$$

convention:

1. We assume function application is left-associative.
   i.e., $(((M N_1) N_2) N_3)$ can be written as $M N_1 N_2 N_3$

2. We assume the scope of λ extend to the end
   i.e. $\lambda z. \lambda x. x y z$ denotes $\lambda z.(\lambda x. x y z)$

3. In "$\lambda x. M$", we call 'x' a "bound variabe".

4. In "$\lambda x. y$", we call 'y' a "free variable"

5. We can allways rename a bound variable to a fresh variable. e.g

$$\cancel{\lambda x y} \quad \lambda x. x \equiv \lambda y. y \equiv \lambda z. z \ \ldots$$

~~fst M | snd M | ()~~

How do we determine $\Gamma \vdash M : A$?

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x . M : A \to B} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash (M, N) : A \times B}$$

$$\frac{}{\Gamma \vdash () : Unit} \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathrm{fst}\, M : A} \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathrm{snd}\, M : B}$$

So to check $\Gamma \vdash M : A$, we can use the above rules.

e.g. $\vdash \lambda x . (\mathrm{snd}\, x, \mathrm{fst}\, x) : A \times B \to B \times A$

$\vdash \lambda x . x : A \to A$, $\vdash \lambda x . () : A \to Unit$

$\vdash \lambda x . x (\lambda y . y) :$

$((A \to A) \to B) \to B$

$$\vdash \lambda x.(x, x) : A \to A \times A$$

$$\vdash \lambda x.\lambda y.\, y : A \to B \to B$$

Evaluation: left-to-right, call-by-value.

values: $V ::= \lambda x.M \mid (V, V') \mid x \mid ()$

$$\frac{}{(\lambda x.M)V \rightsquigarrow [V/x]M}$$

$$\frac{}{\mathrm{fst}(V_1, V_2) \rightsquigarrow V_1} \qquad \frac{}{\mathrm{snd}(V_1, V_2) \rightsquigarrow V_2}$$

$$\frac{M \rightsquigarrow M'}{MN \rightsquigarrow M'N} \qquad \frac{M \rightsquigarrow M'}{V_1 M \rightsquigarrow V_1 M'}$$

$$\frac{M \rightsquigarrow M'}{\mathrm{fst}\, M \rightsquigarrow \mathrm{fst}\, M'} \qquad \frac{M \rightsquigarrow M'}{\mathrm{snd}\, M \rightsquigarrow \mathrm{snd}\, M'}$$

$$\frac{M \rightsquigarrow M'}{(M, N) \rightsquigarrow (M', N)} \qquad \frac{N \rightsquigarrow N'}{(V_1, N) \rightsquigarrow (V_1, N')}$$

Notation, we write $M \overset{*}{\rightsquigarrow} V$, if $\underbrace{M \rightsquigarrow M_1 \rightsquigarrow M_2 \cdots}_{\text{finite steps}} \rightsquigarrow V$