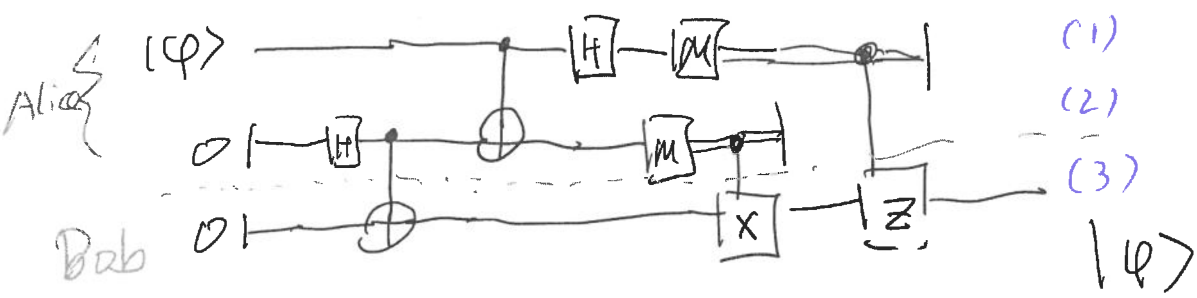
# * Teleportation Circuit




Teleportation protocol:

How would we verify this claim?

$$|\varphi\, 0\, 0\rangle \xmapsto{\beta} \frac{1}{\sqrt{2}}\left(|\varphi\rangle|00\rangle + |\varphi\rangle|11\rangle\right)$$

Let $|\varphi\rangle = a|0\rangle + b|1\rangle$

$$= \frac{1}{\sqrt{2}}\left(a|000\rangle + b|100\rangle + a|011\rangle + b|111\rangle\right)$$

$$\xmapsto{CNOT_{1\to 2}} \frac{1}{\sqrt{2}}\left(a|000\rangle + b|110\rangle + a|011\rangle + b|101\rangle\right)$$

$$\xmapsto{H_1} \frac{1}{\sqrt{2}}\left(a|+\rangle|00\rangle + b|-\rangle|10\rangle + a|+\rangle|11\rangle + b|-\rangle|01\rangle\right)$$

$$= \frac{1}{2}\left(a|\underline{00}0\rangle + a|\underline{10}0\rangle + b|\underline{01}0\rangle - b|110\rangle + a|\underline{01}1\rangle + a|\underline{11}1\rangle + b|\underline{00}1\rangle - b|101\rangle\right)$$

$$
\begin{aligned}
= \frac{1}{2}\big( \; & |00\rangle (a|0\rangle + b|1\rangle) + \\
& |01\rangle (a|1\rangle + b|0\rangle) + \\
& |10\rangle (a|0\rangle - b|1\rangle) + \\
& |11\rangle (a|1\rangle - b|0\rangle) \; \big)
\end{aligned}
$$

So. $M_1 = M_2 = 0$. then we are don.

if $M_1 = 0$, $M_2 = 1$, then we apply $X$ gate to correct.

if $M_1 = 1$, $M_2 = 0$, then we apply $Z$ gate to correct.

otherwise, we apply $-\boxed{X}\boxed{Z}-$ gate.

All these happen with equal probability.

* Principle of ~~deffer~~ deferred measurement.

Measurements done in the middle of the circuit can be deferred to the end.

In particular, we have the following identity.


How to show this?

Suppose we have an input state

$\varphi = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$.

LHS: (left hand side)

$$\varphi \xrightarrow[0]{|a|^2+|b|^2} a|0\rangle + b|1\rangle$$

$$\varphi \xrightarrow[1]{|c|^2+|d|^2} c|0\rangle + d|1\rangle \xrightarrow{U} cU|0\rangle + dU|1\rangle.$$

So LHS = RHS.

RHS: (right hanside)

$$\varphi \xrightarrow{CU} a|00\rangle + b|01\rangle + |1\rangle cU|0\rangle + |1\rangle dU|1\rangle$$

$$\xrightarrow[0]{|a|^2+|b|^2} a|0\rangle + b|1\rangle$$

$$\xrightarrow{|c|^2+|d|^2} cU|0\rangle + dU|1\rangle.$$

* Construction of Toffoli gate.

Recall that Clifford gate set is

$\{ S, H, CNOT \}$.

And $T|0\rangle = |0\rangle$
$T|1\rangle = e^{i\frac{\pi}{4}}|1\rangle$.

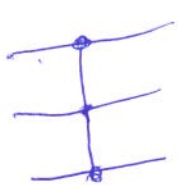
or $T|a\rangle = e^{i\frac{\pi}{4}a}|a\rangle$, where $a \in \{0,1\}$.

So $T$ moves the "state" $a$ into the "Phase" $e^{i\frac{\pi}{4}a}$.

$CNOT|a,b\rangle = |a, a\oplus b\rangle$, $a,b \in \{0,1\}$.

we have.


similarly, we have

is CCZ gate (control-controled-Z).

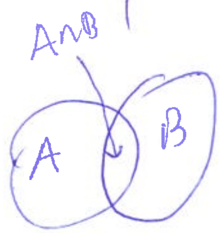
$CCZ|a,b,c\rangle = e^{i\pi(a\cdot b\cdot c)}|a,b,c\rangle$

Note that $e^{i\pi} = -1$.

So to construct Toffoli gate using Clifford+T, we can just construct CCZ using Clifford + T.

* Inclusion-exclusion principle. (For set).

$$|A \cap B| = |A| + |B| - |A \cup B|$$




For binary number, we have:

$$2ab = a + b - a \oplus b$$

$$\text{So } 4abc = 2(2ab)c = 2(a+b-a\oplus b)\cdot c$$

$$= 2ac + 2bc - 2(a\oplus b)c$$

$$= a+c-a\oplus c + b + c - b\oplus c - (a\oplus b + c - a\oplus b\oplus c)$$

$$= a+b+c - a\oplus c - b\oplus c - a\oplus b + a\oplus b\oplus c$$

$$\text{So } e^{i\pi(abc)} = e^{i\frac{\pi}{4}(4abc)} = e^{i\frac{\pi}{4}(a+b+c-a\oplus c-b\oplus c-a\oplus b+a\oplus b\oplus c)}$$

$$= e^{i\frac{\pi}{4}a}\cdot e^{i\frac{\pi}{4}b}\cdot e^{i\frac{\pi}{4}c}\cdot e^{-i\frac{\pi}{4}(a\oplus c)}\cdot e^{-i\frac{\pi}{4}(b\oplus c)}\cdot e^{-i\frac{\pi}{4}(a\oplus b)}\cdot e^{i\frac{\pi}{4}(a\oplus b\oplus c)}$$