2 qubits.

$$\text{Qubit} = \{ a|0\rangle + b|1\rangle \mid a,b \in \mathbb{C},\ |a|^2+|b|^2=1 \}$$

$$\text{Qubit} \otimes \text{Qubit}$$

$$= \{ a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \mid |a|^2+|b|^2+|c|^2+|d|^2=1 \}.$$

if $|\varphi_1\rangle \in \text{Qubit}$

$|\varphi_2\rangle \in \text{Qubit}$

$|\varphi_1\rangle \otimes |\varphi_2\rangle = |\varphi_1 \varphi_2\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$

Note that the converse is not true. if $\psi \in \text{Qubit} \otimes \text{Qubit}$.

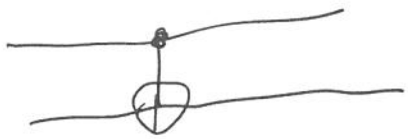
ψ need not be separable, i.e.

$\cancel{\varphi}\ \psi = |\varphi_1\rangle \otimes |\varphi_2\rangle$,

e.g. $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ can not be written

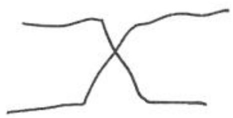
as $|\varphi_1\rangle \otimes |\varphi_2\rangle$.

** controlled-not gate



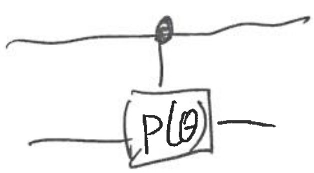
$$\text{CNOT}|00\rangle = |00\rangle$$
$$\text{CNOT}|01\rangle = |01\rangle$$
$$\rightarrow \text{CNOT}|10\rangle = |11\rangle$$
$$\rightarrow \text{CNOT}|11\rangle = |10\rangle$$

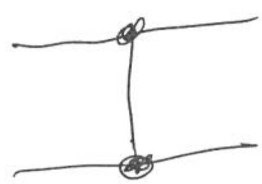
** swap gate



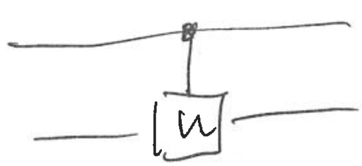
$$\text{swap}|ab\rangle = |ba\rangle.$$

** controlled Phased gate. CZ gate

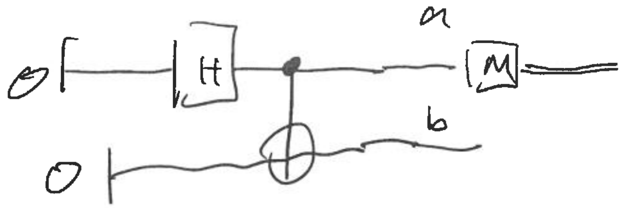




** if U is unitary, CU is also unitary.



* A few more common circuits.



$$|00\rangle \xmapsto{H} |+\rangle|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle$$

$$= \frac{1}{\sqrt{2}}(|00\rangle+|10\rangle)$$

$$\xmapsto{CNOT_{1\to2}} \quad \underline{\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)}$$

bell state β_{00}.

observation: 1. if we measure a or b.
we ~~kn~~ will know the value of the other.

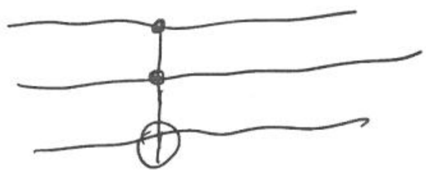
2. $\{\beta_{00}, \beta_{01}, \beta_{11}, \beta_{10}\}$ also forms a basis for 2-qubit state.

i.e.

$$|\varphi\rangle = a\beta_{00} + b\beta_{01} + c\beta_{10} + d\beta_{11}$$

$$|a|^2+|b|^2+|c|^2+|d|^2=1$$

* Toffoli gate. CCX gate



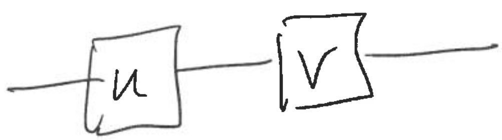
$CCX|11a\rangle = CCX|11\bar{a}\rangle$

otherwise

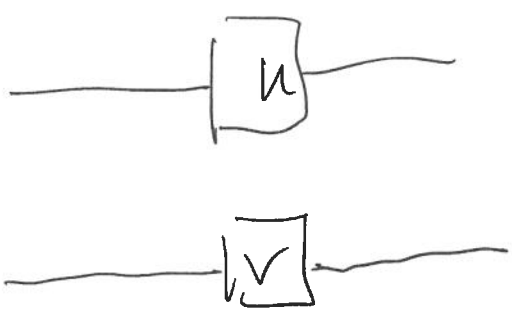
$CCX|\varphi_1 \varphi_2 \varphi_3\rangle = |\varphi_1, \varphi_2, \varphi_3\rangle$

or $CCX|xyz\rangle = |x, y, (x\cdot y)\oplus z\rangle$

* We can compose gates horizontally and vertically.

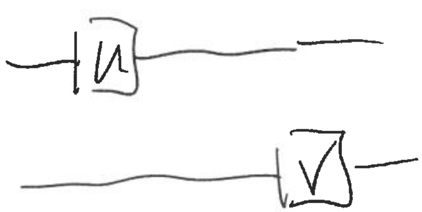


$V \circ U$

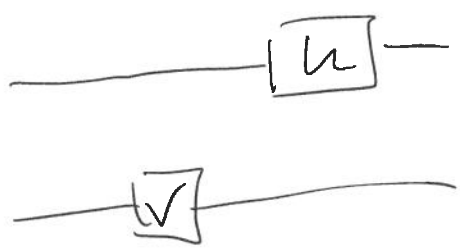


$U \otimes V$

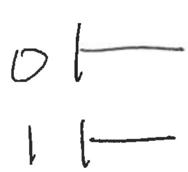
=



=

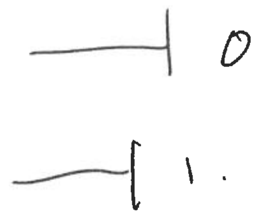


* initialization



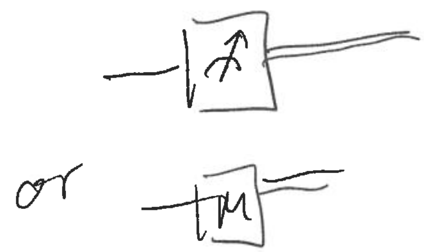
prepare a qubit in $|0\rangle$ or $|1\rangle$ state.

*. Termination



"assert" a qubit is in $|0\rangle$ or $|1\rangle$ state and then discard.

* Measurement



Qubit $\rightarrow$ Bit.

$M(a|0\rangle + b|1\rangle)$

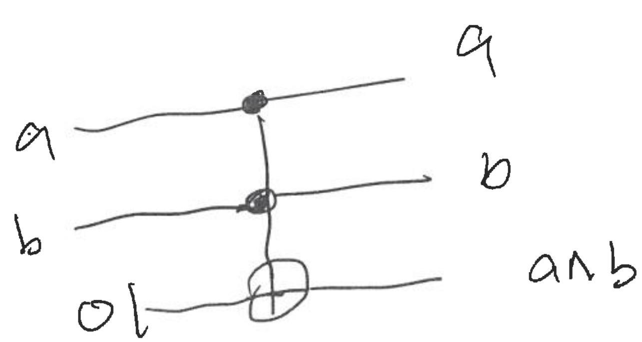
returns $|0\rangle$ with prob $|a|^2$.

$|1\rangle$ with prob $|b|^2$.

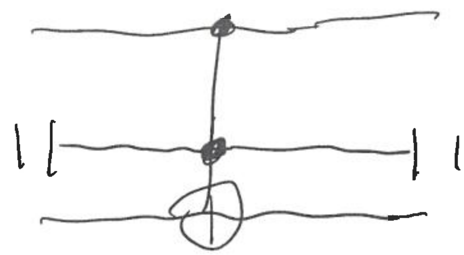
*. Classical circuits.

Toffoli and Init/Term are enough for any classical/reversible circuits!

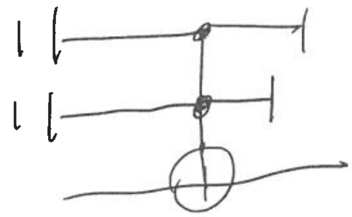
1. Conjunction



2. CNOT from Toffoli



3. Not gate

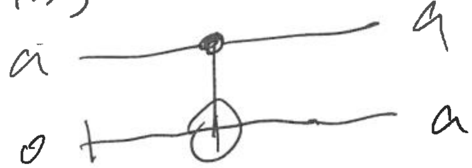


4. Disjunction

$$a \vee b = \neg\neg a \vee \neg\neg b$$
$$= \neg(\neg a \wedge \neg b)$$

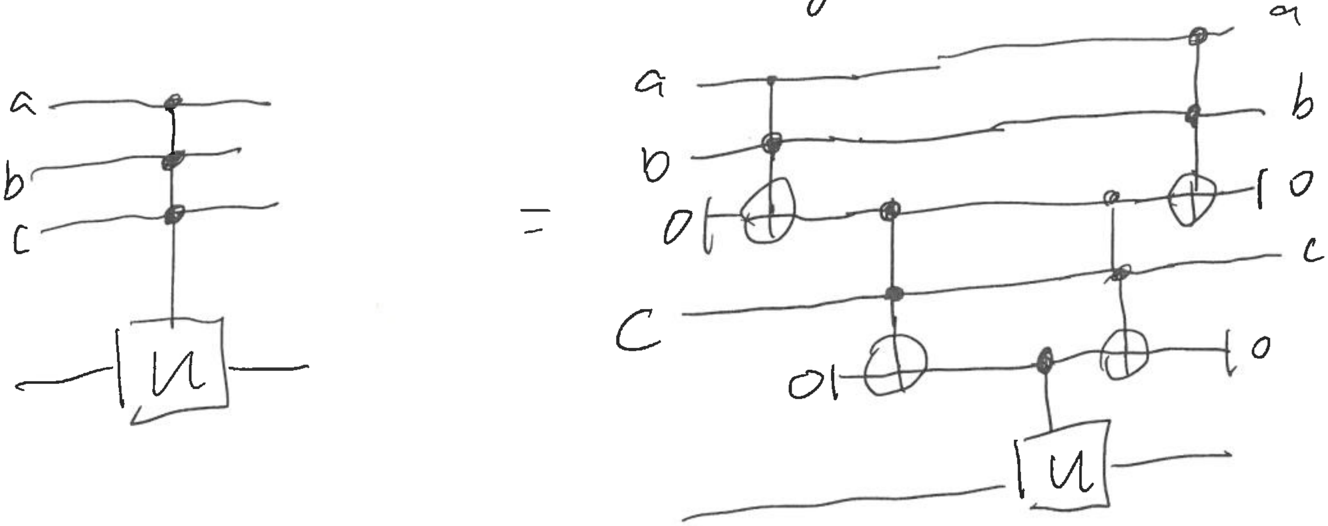
5. Classical copying

$a \in \{|0\rangle, |1\rangle\}$



only when a is classical.
i.e. $a = |0\rangle$ or $a = |1\rangle$

* Multi Controlled unitary



This generalizes to n-fold controls.

* Some useful facts and terminolog.

$\{I, X, Y, Z, S, H, CNOT\}$

Clifford gate set~~s~~

<u>Thm</u>: Quantum circuit consisted of only Clifford gate sets and standard measurement can be efficiently implemented by Classical computer.

To achieve quantum advantage, one often work with Clifford + T gate set.

Def. A gate set is <u>universal</u> for quantum if Any n-Qubit unitary operation can be implemented within the gate set.

<u>Thm</u>. CNOT with 1-Qubit unitary is universal.

Approximating unitary.

Def. Suppose V is a unitary construced from a finite gate set.
~~We say~~ Suppose U is a unitary.
we say V approximate U up to $\varepsilon > 0$
if $\|U - V\| \leq \varepsilon$.