Quantum Programming Languages
CSCE 790 Section 008 Homework 3
Due: February 23, Friday, 2pm

Recall that we have the following typing rules for Linear typed Lambda Calculus.

$$\frac{}{\Phi, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \multimap B}$$

$$\frac{\Phi, \Gamma_1 \vdash M : A \multimap B \quad \Phi, \Gamma_2 \vdash N : A}{\Phi, \Gamma_1, \Gamma_2 \vdash MN : B} \qquad \frac{\Phi, \Gamma_1 \vdash M : A \quad \Phi, \Gamma_2 \vdash N : B}{\Phi, \Gamma_1, \Gamma_2 \vdash (M, N) : A \otimes B}$$

$$\frac{\Phi, \Gamma_1 \vdash M : A \otimes B \quad \Phi, \Gamma_2, x : A, y : B \vdash N : C}{\Phi, \Gamma_1, \Gamma_2 \vdash \mathbf{let}\ (x, y) = M\ \mathbf{in}\ N : C} \qquad \frac{\Phi \vdash M : A}{\Phi \vdash \mathbf{lift} M : !A}$$

$$\frac{\Gamma \vdash M : !A}{\Gamma \vdash \mathbf{force} M : A} \qquad \frac{}{\Phi \vdash () : \mathbf{Unit}}$$

$$\frac{}{\Phi \vdash \mathbf{True} : \mathbf{Bool}} \qquad \frac{}{\Phi \vdash \mathbf{False} : \mathbf{Bool}}$$

$$\frac{\Phi, \Gamma_1 \vdash M : \mathbf{Bool} \quad \Phi, \Gamma_2 \vdash N_1 : C \quad \Phi, \Gamma_2 \vdash N_2 : C}{\Phi, \Gamma_1, \Gamma_2 \vdash \mathbf{if}\, M\, \mathbf{then}\, N_1\, \mathbf{else}\, N_2 : C}$$

Note that parameter types are $P ::= \mathbf{Unit} \mid \mathbf{Bool} \mid P \otimes P' \mid !A$ and Φ is a parameter context, i.e., $\Phi = x_1 : P_1, ..., x_n : P_n$ (when $n = 0$, we have an empty parameter context).

1. Consider the following closed lambda terms (i.e., they do not contain free variables). Determine if the following typing judgment are valid. If a typing judgment is valid, give a typing derivation using the typing rules specified above. If not, explain why it is not valid.

   (a) (2 points) $\vdash \lambda x.() : \mathbf{Qubit} \multimap \mathbf{Unit}$

   (b) (2 points) $\vdash \lambda x.() : \mathbf{Bool} \multimap \mathbf{Unit}$

   (c) (2 points) $\vdash \lambda x.\mathbf{lift}\, x : \mathbf{Qubit} \multimap !\mathbf{Qubit}$.

   (d) (2 points) $\vdash \lambda z.\mathbf{let}\, (x, y) = z\ \mathbf{in}\ y : \mathbf{Qubit} \otimes \mathbf{Qubit} \multimap \mathbf{Qubit}$.

   (e) (2 points) $\vdash \lambda z.\mathbf{let}\, (x, y) = z\ \mathbf{in}\ y : \mathbf{Bool} \otimes \mathbf{Qubit} \multimap \mathbf{Qubit}$.

   (f) (2 points) $\vdash \lambda x.\lambda z.\mathbf{if}\, z\, \mathbf{then}\, x\, \mathbf{else}\, x : \mathbf{Qubit} \multimap \mathbf{Bool} \multimap \mathbf{Qubit}$.

   (g) (2 points) $\vdash \lambda x.\lambda y.\lambda z.\mathbf{if}\, z\, \mathbf{then}\, y\, \mathbf{else}\, z : \mathbf{Qubit} \multimap \mathbf{Qubit} \multimap \mathbf{Bool} \multimap \mathbf{Qubit}$.

2. Type inhabitation problem is the problem of finding a term for a given type under the empty typing context. Consider the following types, determine if there is a term that inhabits the type. If a type is inhabitable, then provide the term and the typing derivation; if a type is not inhabitable, explain why.

   (a) (2 points) $!A \multimap A$

(b) (2 points) $A \multimap !A$

(c) (2 points) $!(A \multimap A)$

(d) (2 points) $!!A \multimap !A$

(e) (2 points) $!A \multimap !!A$

(f) (2 points) $!A \otimes !B \multimap !(A \otimes B)$

(g) (2 points) $!(A \otimes B) \multimap !A \otimes !B$

(h) (2 points) $!(A \multimap B) \multimap (!A \multimap B)$

(i) (2 points) $(!A \multimap B) \multimap !(A \multimap B)$

(j) (2 points) $(A \otimes B \multimap C) \multimap (A \multimap B \multimap C)$

(k) (2 points) $(A \multimap B \multimap C) \multimap (A \otimes B \multimap C)$