Quantum Programming Languages
CSCE 790 Section 008 Homework 2
Due: Feburary 5, Monday, 2pm

Recall that we have the following typing rules for Simply Typed Lambda Calculus with sums and products.

$$\frac{(x:A)\in\Gamma}{\Gamma\vdash x:A} \qquad \frac{\Gamma, x:A\vdash M:B}{\Gamma\vdash \lambda x.M:A\to B} \qquad \frac{\Gamma\vdash M:A\to B \quad \Gamma\vdash N:A}{\Gamma\vdash MN:B}$$

$$\frac{\Gamma\vdash M:A \quad \Gamma\vdash N:B}{\Gamma\vdash (M,N):A\times B} \qquad \frac{\Gamma\vdash M:A\times B}{\Gamma\vdash \mathbf{fst}(M):A} \qquad \frac{\Gamma\vdash M:A\times B}{\Gamma\vdash \mathbf{snd}(M):B}$$

$$\frac{\Gamma\vdash M:A}{\Gamma\vdash \mathbf{left}(M):A+B} \qquad \frac{\Gamma\vdash M:B}{\Gamma\vdash \mathbf{right}(M):A+B} \qquad \frac{}{\Gamma\vdash ():\mathbf{Unit}}$$

$$\frac{\Gamma\vdash M:\bot}{\Gamma\vdash \mathbf{abort}(M):A} \qquad \frac{\Gamma\vdash M:A+B \quad \Gamma,x:A\vdash N_1:C \quad \Gamma,y:B\vdash N_2:C}{\Gamma\vdash \mathbf{case}(M)\mathbf{of}\{\mathbf{left}(x)\to N_1;\mathbf{right}(y)\to N_2\}:C}$$

1. Consider the following closed lambda terms (i.e., they do not contain free variables). Determine if they are typable under the empty typing context. If a term is typable, give it a type and its typing derivation using the typing rules specified above. If not, explain why it is not typable. For example, the closed lambda term $\lambda x.x$ is typable with a type like $A \to A$, and here is its typing derivation.

$$\frac{\overline{x:A\vdash x:A}}{\vdash \lambda x.x:A\to A}$$

   (a) (2 points) $\lambda x.x(\lambda y.y)$
   (b) (2 points) $\lambda x.xx$
   (c) (2 points) $\mathbf{fst}(\mathbf{left}(()))$
   (d) (2 points) $\mathbf{abort}(\lambda x.x)$
   (e) (2 points) $\lambda x.\mathbf{case}(x)\mathbf{of}\{\mathbf{left}(y)\to y(\lambda x.x);\mathbf{right}(z)\to z\}$

2. Type inhabitation problem is the problem of finding a term for a given type under the empty typing context. For example, there is a term that inhabits the type $A \to A$, e.g, we have $\vdash \lambda x.x : A \to A$. Whereas there is no term that inhabits the empty type $\bot$, i.e., we can not find a term M such that $\vdash M : \bot$.

   Consider the following types, determine if there is a term that inhabits the type. If a type is inhabitable, then provide the term and the typing derivation; if a type is not inhabitable, explain why.

   (a) (2 points) $A \times B \to B \times A$
   (b) (2 points) $A \to B \to (A \times B)$
   (c) (2 points) $(A \to B) \to (B \to C) \to (A \to C)$

(d) (2 points) $(A \to C) \to C$

(e) (2 points) $((A \to \bot) + B) \to (A \to B)$

(f) (2 points) $(A \to B) \to ((A \to \bot) + B)$

3. (2 points) Use left-to-right, call-by-value reduction to reduce the lambda term $(\lambda x.y)((\lambda z.zz)(\lambda w.w))$.

4. (3 points) Consider the lambda term $N := \lambda x.\lambda y.y(xxy)$. Define $\Theta := NN = (\lambda x.\lambda y.y(xxy))(\lambda x.\lambda y.y(xxy))$. Show that for any lambda term V that is a value, we have $\Theta V \rightsquigarrow^* V(\Theta V)$, i.e., ΘV is a fixpoint of V. The term Θ is called *Turing's fixpoint*. Is Θ typable? (A term M is typable if there is a type A such that $\vdash M : A$).