2/6/2025

Evaluation. $\Gamma \vdash M : A$.

"Left-to-Right, call-by-value" evaluation.

$$M ::= c \mid x \mid \lambda x.M \mid M\,N \mid (M, N) \mid \mathrm{fst}\,M \mid \mathrm{snd}\,M.$$

Values: $V ::= c \mid \lambda x.M \mid (V_1, V_2)$

Evaluation relation: $\underbrace{M \rightsquigarrow M_1}$ $\rightsquigarrow \cdots\cdots M_n \rightsquigarrow V$

evaluating a term M to a value V.

$\boxed{M \rightsquigarrow M'}$ small step reduction

Remark: "call-by-name"

$$\beta \ \frac{}{(\lambda x.M)\,N \rightsquigarrow [N/x]M}.$$

$$\beta \ \frac{}{(\lambda x.M)\,V \rightsquigarrow \underbrace{[V/x]M}}.$$

β-reduction. value! means substituting all the variable x in M, by the value V.

$$\mathrm{fst} \ \frac{}{\mathrm{fst}(V_1, V_2) \rightsquigarrow V_1} \qquad \mathrm{snd} \ \frac{}{\mathrm{snd}(V_1, V_2) \rightsquigarrow V_2}.$$

congruence rules:

$$\frac{M \rightsquigarrow M'}{M\,N \rightsquigarrow M'\,N}. \qquad \frac{N \rightsquigarrow N'}{V\,N \rightsquigarrow V\,N'}$$

$$\frac{M \rightsquigarrow M'}{\mathrm{fst}\ M \rightsquigarrow \mathrm{fst}\ M'} \qquad \frac{M \rightsquigarrow M'}{\mathrm{snd}\ M \rightsquigarrow \mathrm{snd}\ M'}$$

$$\left\{ \frac{M \rightsquigarrow M'}{(M, N) \rightsquigarrow (M', N)} \qquad \frac{N \rightsquigarrow N'}{(V, N) \rightsquigarrow (V, N')} \right.$$

example :

$$\dfrac{\dfrac{\beta\ \dfrac{}{(\lambda x.x)\mathrm{true} \rightsquigarrow \mathrm{true}}}{((\lambda x.x)\mathrm{true}, (\lambda y.y)\mathrm{false}) \rightsquigarrow (\mathrm{true}, (\lambda y.y)\mathrm{false})}}{(\lambda x.(\mathrm{snd}\ x, \mathrm{fst}\ x))\underbrace{((\lambda x.x)\mathrm{true}, (\lambda y.y)\mathrm{false})}_{: \mathrm{Bool} \times \mathrm{Bool}} \rightsquigarrow (\lambda x.(\mathrm{snd}\ x, \mathrm{fst}\ x))(\mathrm{true}, (\lambda y.y)\mathrm{false})}$$

$$\dfrac{\dfrac{\beta\ \dfrac{}{(\lambda y.y)\mathrm{false} \rightsquigarrow \mathrm{false}}}{(\mathrm{true}, (\lambda y.y)\mathrm{false}) \rightsquigarrow (\mathrm{true}, \mathrm{false})}}{(\lambda x.(\mathrm{snd}\ x, \mathrm{fst}\ x))(\mathrm{true}, (\lambda y.y)\mathrm{false}) \rightsquigarrow (\lambda x.(\mathrm{snd}\ x, \mathrm{fst}\ x))(\mathrm{true}, \mathrm{false})}$$

$$\beta\ \dfrac{}{(\lambda x.(\mathrm{snd}\ x, \mathrm{fst}\ x))(\mathrm{true}, \mathrm{false}) \rightsquigarrow (\mathrm{snd}(\mathrm{true}, \mathrm{false}), \mathrm{fst}(\mathrm{true}, \mathrm{false}))}$$

$$\frac{}{\text{snd}(\text{true}, \text{false}) \rightsquigarrow \text{false}}\ \text{snd}$$

$$\frac{\text{snd}(\text{true}, \text{false}) \rightsquigarrow \text{false}}{(\text{snd}(\text{true}, \text{false}), \text{fst}(\text{true}, \text{false})) \rightsquigarrow (\text{false}, \text{fst}(\text{true}, \text{false}))}$$

$$\frac{}{\text{fst}(\text{true}, \text{false}) \rightsquigarrow \text{true}}\ \text{fst}$$

$$\frac{\text{fst}(\text{true}, \text{false}) \rightsquigarrow \text{true}}{(\text{false}, \text{fst}(\text{true}, \text{false})) \rightsquigarrow \underline{(\text{false}, \text{true})}} \quad : \text{Bool} \times \text{Bool}.$$

* Type preservation property.

If $\Gamma \vdash M : A$, and $\underbrace{M \overset{*}{\rightsquigarrow} V}_{\text{shorthand for}}$,

$M \rightsquigarrow M_1 \rightsquigarrow M_2 \cdots \rightsquigarrow V$.

then $\Gamma \vdash V : A$.

* Some additions to simply typed Lambda calculus.

① Unit type.

$A ::= \ldots\ldots \mid \text{Unit}$

$M ::= \ldots\ldots \mid ()$

$$\frac{}{\Gamma \vdash () : \text{Unit}}$$

$V ::= \ldots \mid ()$

$\mathrm{Unit} \to \mathrm{Bool}$.

$\mathrm{Bool} \to \mathrm{Unit}$.

$\mathrm{initialize} : \mathrm{Unit} \to \mathrm{Qubit}$.

for later.

$$\boxed{(\mathrm{Unit} \to A) \cong A}$$

informal.

* Sum type.

$$A ::= \cdots \mid A + B$$

$$M ::= \cdots \mid \mathrm{left}\ M \mid \mathrm{Right}\ M$$

$$\mid \mathrm{case}\ M\ \{ \mathrm{left}\ x \to N_1 ;\ \mathrm{Right}\ x \to N_2 \}.$$

Typing rules:

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{left}\ M : A + B} \qquad \frac{\Gamma \vdash M : B}{\Gamma \vdash \mathrm{right}\ M : A + B}.$$

$$\frac{\Gamma \vdash M : A+B \qquad \Gamma, x : A \vdash N_1 : C \qquad \Gamma, x : B \vdash N_2 : C}{\Gamma \vdash \mathrm{case}\ M\ \mathrm{of}\ \{ \mathrm{left}\ x \to N_1 ;\ \mathrm{right}\ x \to N_2 \} : C}$$

$V ::= \cdots \mid \text{left } V \mid \text{Right } V$

$$\frac{}{\text{case } (\text{left } V) \text{ of } \{\text{left } x \to N_1 \,;\, \text{right } x \to N_2\} \rightsquigarrow [V/x]N_1}$$

$$\frac{}{\text{case } (\text{right } V) \text{ of } \{\text{left } x \to N_1 \,;\, \text{right } x \to N_2\} \rightsquigarrow [V/x]N_2}$$

$$\frac{M \rightsquigarrow M'}{\text{case } M \text{ of } \{\cdots\} \rightsquigarrow \text{case } M' \text{ of } \{\cdots\}}$$

e.g. $\text{Bool} = \text{Unit} + \text{Unit}$

$\text{true} = \text{left } ()$

$\text{false} = \text{right } ()$

$$\text{ife } M \; N_1 \; N_2$$

$$= \text{case } M \text{ of}$$

$$\text{left } x \to N_1$$

$$\text{right } x \to N_2.$$

x is fresh.

$x \notin N_1$ or N_2.

$$\text{ife true } N_1 \; N_2 \rightsquigarrow N_1$$

$$\parallel$$

$$\text{case left()} \text{ of}$$

$$\{ \text{left } x \to N_1; \quad \rightsquigarrow [()/x] N_1$$

$$\text{right } x \to N_2 \} \quad = N_1.$$

$$\text{ife false } N_1 \; N_2 \rightsquigarrow N_2.$$

$$A + B = \text{Unit} + \text{Unit} + \text{Unit}$$

$$= (\text{Unit} + \text{Unit}) + \text{Unit}$$

$$\cong \text{Unit} + (\text{Unit} + \text{Unit})$$

equivalence but not equal.

* Empty type

$$A ::= \cdots \mid \bot$$

$$M ::= \cdots \mid \text{abort}_A M$$

$$\frac{\Gamma \vdash M : \bot}{\Gamma \vdash \text{abort}_A M : A}$$