1/30/2025

Simply Typed Lambda Calculus.

$\boxed{\Gamma \vdash M : A}$ ← Type

Typing judgment.

Typing environment.

programs/terms

Types: $A, B ::= \text{Bool} \mid \text{Nat} \mid A \times B \mid A \to B.$

BNF

"Backus-Naur-Form."

examples of types:

$\text{Bool} \times \text{Nat}$, $\text{Bool} \to \text{Nat}$

$(\text{Bool} \to \text{Bool}) \to (\text{Nat} \to \text{Nat})$

"higher-order types":

$(A \to A) \to (A \to A)$

* Product is left associative.

Arrow is right associative.

$A \times B \times C = (A \times B) \times C \neq A \times (B \times C)$

$A \to B \to C = A \to (B \to C) \neq (A \to B) \to C.$

$(A \to B) \times (C \to D)$

$(A \times B) \to (C \times D)$

lambda term. "anonymous function" ↘ Function application ↓

* Terms. $M ::= x \mid (M, N) \mid \lambda x. M \mid \underline{M N}$

"BNF for terms" $\mid c \mid \underbrace{\mathrm{fst}\ M \mid \mathrm{snd}\ M}$ first/second projection of a pair.

"constant" e.g. true, false.

examples of terms.

"identity function" → $\lambda \underbrace{x}_{\text{binder}}. x = \lambda y. y \neq \lambda y. x$

$\lambda p. (\mathrm{snd}\ p, \mathrm{fst}\ p)$

$\lambda f. \lambda x. f(f x) = \lambda f. (\lambda x. f(f x))$

* Application is left associative.

i.e. $M N_1 N_2 = (M N_1) N_2$.

* Typing environment $\Gamma = x_1 : A_1, x_2 : A_2 \cdots x_n : A_n$.

* Typing rules.

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\ (\mathrm{var}) \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.\underbrace{M}_{\text{body of the } \lambda\text{-expression}} : A \to B}\ (\lambda)$$

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\ (\mathrm{app})$$

$$D \to (A \to B) \to C$$

$$= D \to ((A \to B) \to C)$$

$$\frac{y : D \vdash M : (A \to B) \to C}{\emptyset \vdash \lambda y.M : D \to ((A \to B) \to C)}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash (M, N) : A \times B}\ (\mathrm{Pair}) \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathrm{fst}\ M : A}\ (\mathrm{fst})$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathrm{snd}\ M : B}\ (\mathrm{snd})$$

* Type checking problem:

Given Γ, M, A,
do we have $\Gamma \vdash M : A$?

"turnstile"

* Type inference problem:

Given Γ, M, what is the type of M ?
i.e. how can we find a type A such that $\Gamma \vdash M : A$.

* Type inhabitation problem:

Given Γ, A, can we find a term M s.t. $\Gamma \vdash M : A$.

*An Example of type checking

comment: $B \overset{?}{=} A$

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\dfrac{f : A \to A \in \Gamma}{\Gamma = f : A \to A,\ x : A \vdash f : A \to A}
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\dfrac{f : A \to A \in \Gamma}{\Gamma \vdash f : A \to A}
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\dfrac{f : A \to A \vdash \lambda x. f(f x) : A \to A}{\emptyset \vdash \underbrace{\lambda f. \lambda x. f(f x}) : (A \to A) \to (A \to A)}\ (\lambda)
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$$

$$\emptyset \vdash \lambda f. \lambda x. f(f x) : (A \to A)$$