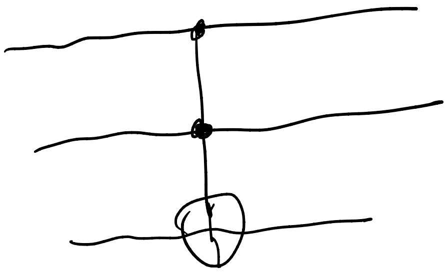
1/28/2025

We want to give a Clifford + T implementation of Toffoli gate.



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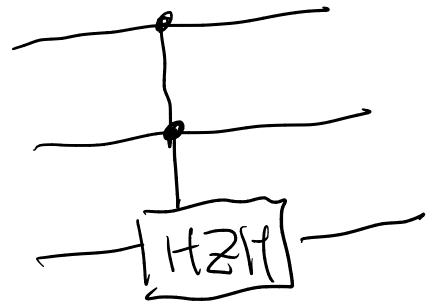

$X = HZH$

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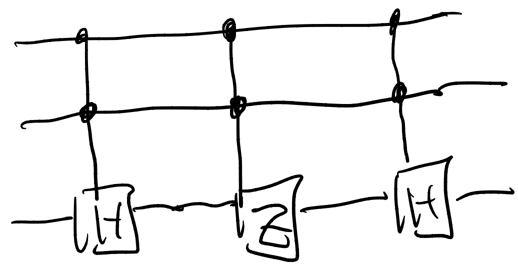

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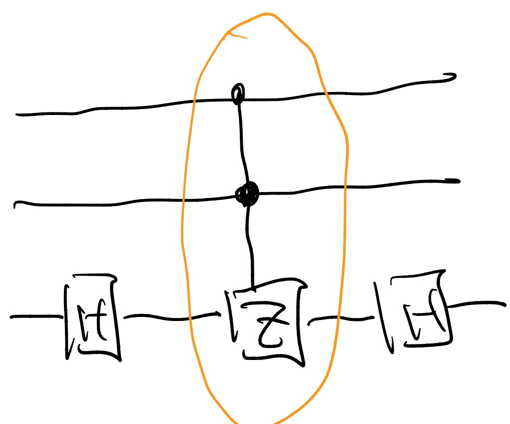
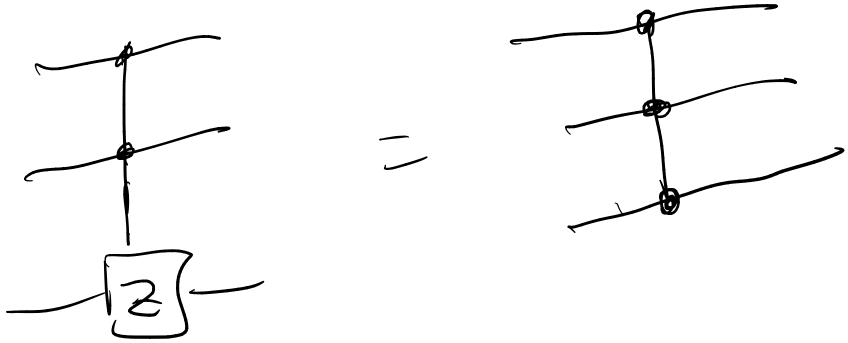


* We just need to give a Clifford + T implementation for CCZ.

$$CCZ|a,b,c\rangle = (-1)^{a \cdot b \cdot c} |a,b,c\rangle$$

$$a, b, c \in \{0, 1\}$$



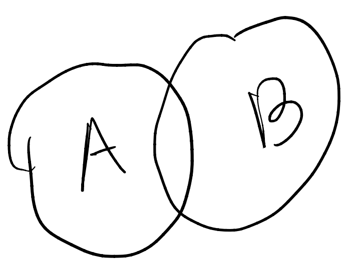
(notation).

$$CCZ\,|a,b,c\rangle = e^{i\pi abc}\,|a,b,c\rangle$$

$$T\,|a\rangle = e^{i\frac{\pi}{4}a}\,|a\rangle \qquad = e^{i\frac{\pi}{4}(4abc)}\,|a,b,c\rangle$$

$$T^{\dagger}|a\rangle = e^{-i\frac{\pi}{4}a}\,|a\rangle$$

* Inclusion-Exclusion principle.

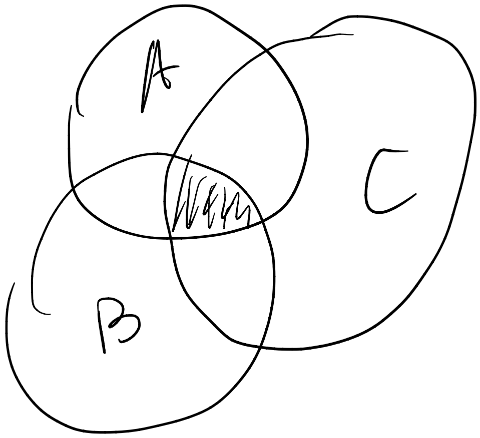


$$|A \cap B| = |A| + |B| - |A \cup B|$$

inclusion-exclusion for binary number

$$\boxed{2a\cdot b = a + b - a \oplus b}$$

①



$$|A \cap B \cap C| = |A| + |B| + |C| - |A \cup B| - |B \cup C| - |A \cup C| + |A \cup B \cup C|$$

②

$$\begin{aligned}
4abc &= 2(2ab)c \\
&= 2(a+b-a\oplus b)\,c \\
&= 2ac + 2bc - 2(a\oplus b)\cdot c \\
&= a+c-a\oplus c+b+c-b\oplus c \\
&\quad -(a\oplus b + c - a\oplus b\oplus c) \\
&= a+c-a\oplus c+b+\cancel{c}-b\oplus c \\
&\quad - a\oplus b - \cancel{c} + a\oplus b\oplus c
\end{aligned}$$

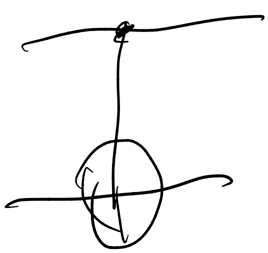
$$\boxed{4abc = a+b+c-a\oplus c-b\oplus c - a\oplus b + a\oplus b\oplus c}$$

$$CCZ\,|a,b,c\rangle = e^{i\pi abc}\,|a,b,c\rangle$$

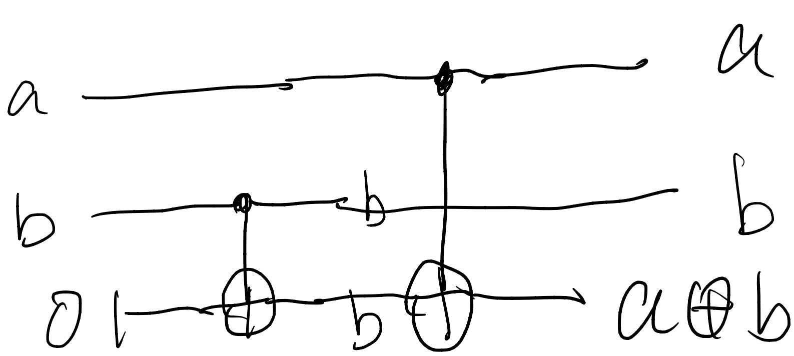
$$= e^{i\frac{\pi}{4}(4abc)}\,|a,b,c\rangle$$

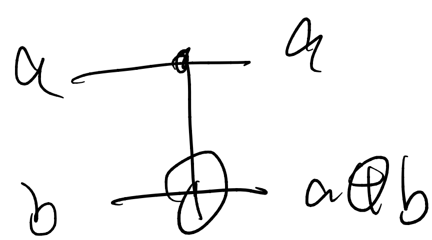
$$= e^{i\frac{\pi}{4}(a+b+c - a\oplus c - b\oplus c - a\oplus b + a\oplus b\oplus c)}\,|a,b,c\rangle$$

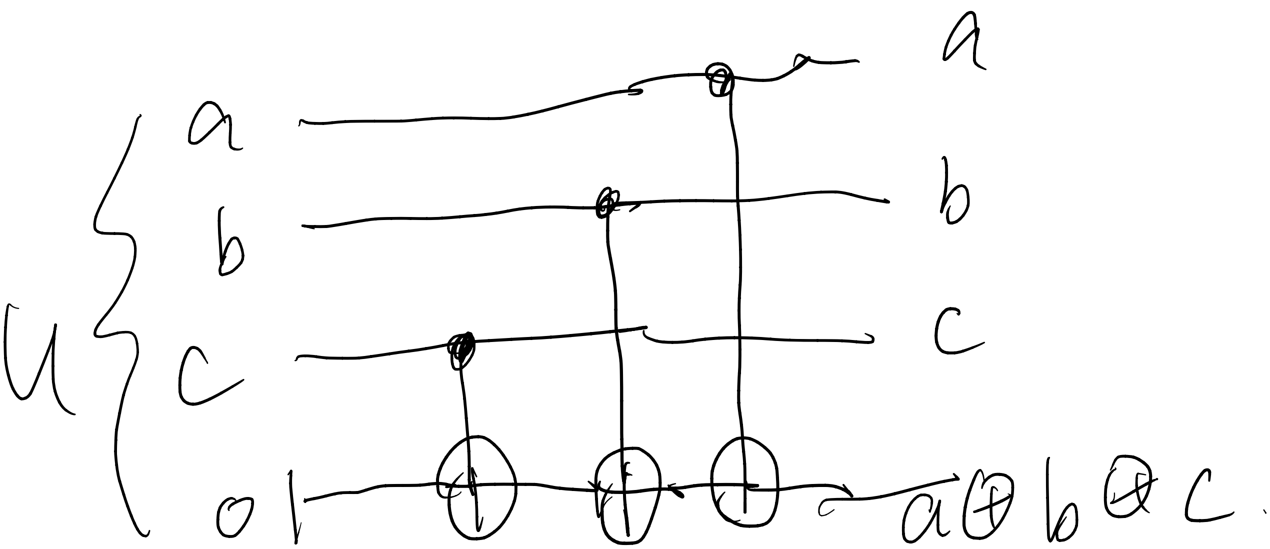
$$= \boxed{e^{i\frac{\pi}{4}a}\cdot e^{i\frac{\pi}{4}b}\cdot e^{i\frac{\pi}{4}c}\cdot e^{-i\frac{\pi}{4}(a\oplus c)} \cdot e^{-i\frac{\pi}{4}(a\oplus b)}\cdot e^{-i\frac{\pi}{4}(b\oplus c)}\cdot e^{i\frac{\pi}{4}(a\oplus b\oplus c)}}\,|a,b,c\rangle$$



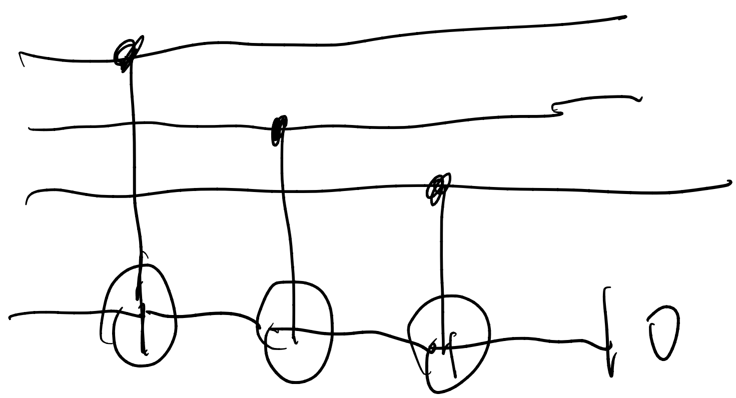
$$CX\,|a,b\rangle = |a, a\oplus b\rangle$$





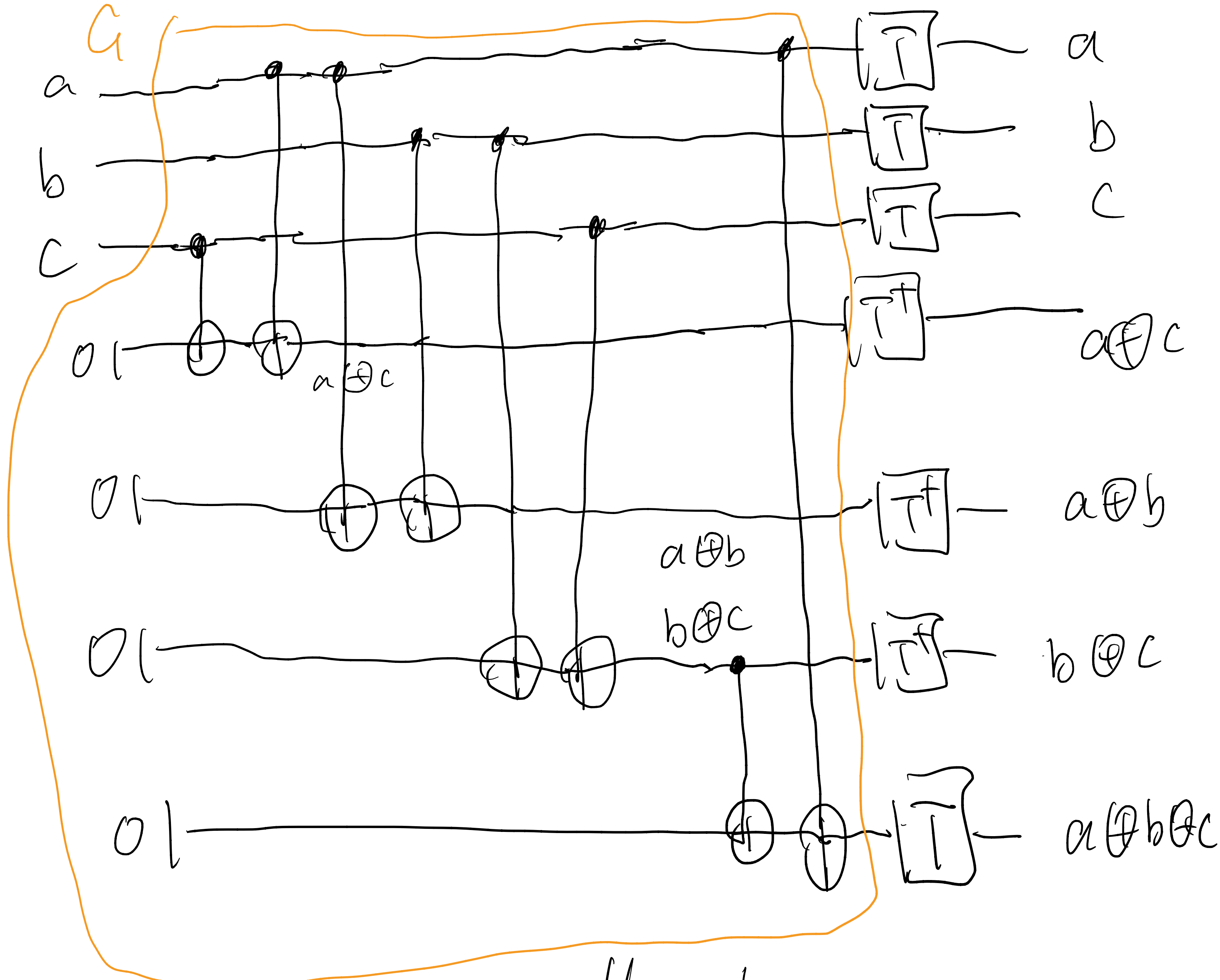


$U^\dagger$



$$|a,b,c\rangle \mapsto e^{i\frac{\pi}{4}a}\cdot e^{i\frac{\pi}{4}b}\cdot e^{i\frac{\pi}{4}c}\cdot \boxed{e^{-i\frac{\pi}{4}(a\oplus c)}}$$

$$e^{-i\frac{\pi}{4}(a\oplus b)}\cdot e^{-i\frac{\pi}{4}(b\oplus c)}\cdot e^{i\frac{\pi}{4}(a\oplus b\oplus c)}\quad |a,b,c\rangle$$

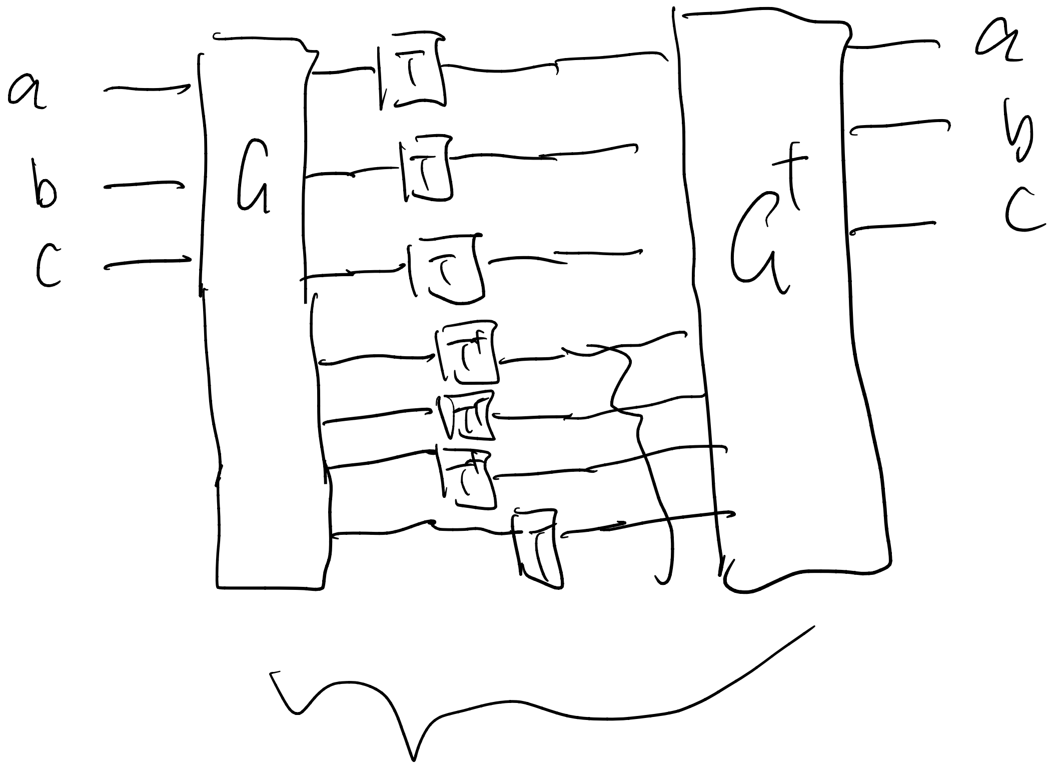


overall phase:

$$e^{i\frac{\pi}{4}a} \cdot e^{i\frac{\pi}{4}b} \cdot e^{i\frac{\pi}{4}c} \cdot e^{-i\frac{\pi}{4}(a\oplus c)}$$

$$e^{-i\frac{\pi}{4}(a\oplus b)} \cdot e^{-i\frac{\pi}{4}(b\oplus c)} \cdot e^{i\frac{\pi}{4}(a\oplus b\oplus c)}$$

$$= (-1)^{abc}$$



* This implementation of CCZ requires 7 T-gates.

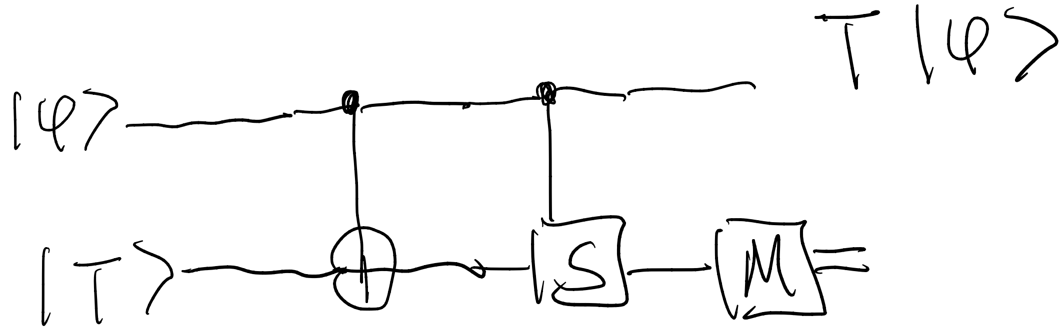
* Current state of the art: CCZ can be implemented with 5 T-gates.

* T gate via "T-state"

$$|T\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle\right)$$

"magic state"

* Circuit



$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

$$|\varphi\rangle \otimes |T\rangle = \left(a|0\rangle + b|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(a|00\rangle + ae^{i\frac{\pi}{4}}|01\rangle + b|10\rangle + be^{i\frac{\pi}{4}}|11\rangle\right)$$

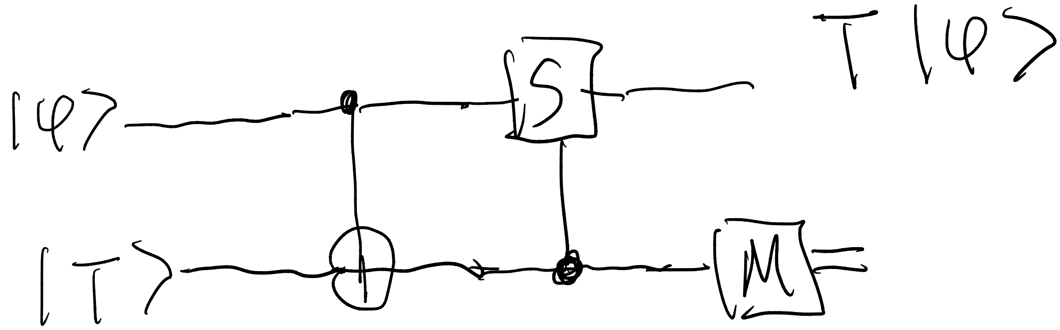
$$\xrightarrow{CX} \frac{1}{\sqrt{2}}\left(a|00\rangle + ae^{i\frac{\pi}{4}}|01\rangle + b|11\rangle + be^{i\frac{\pi}{4}}|10\rangle\right)$$

$$\xrightarrow{CS} \frac{1}{\sqrt{2}}\left(a|00\rangle + ae^{i\frac{\pi}{4}}|01\rangle + be^{i\frac{\pi}{2}}|11\rangle + be^{i\frac{\pi}{4}}|10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(\underbrace{(a|0\rangle + be^{i\frac{\pi}{4}}|1\rangle)}_{T|\varphi\rangle} \otimes |0\rangle + \underbrace{(ae^{i\frac{\pi}{4}}|0\rangle + be^{i\frac{\pi}{2}}|1\rangle)}_{T|\varphi\rangle \simeq e^{i\frac{\pi}{4}}(a|0\rangle + be^{i\frac{\pi}{4}}|1\rangle)} \otimes |1\rangle\Big)$$

* $|\phi\rangle$ is equivalent to $|\psi\rangle$ up to a "global phase" c :

if $|\psi\rangle = c|\phi\rangle$, $|c|^2 = 1$.



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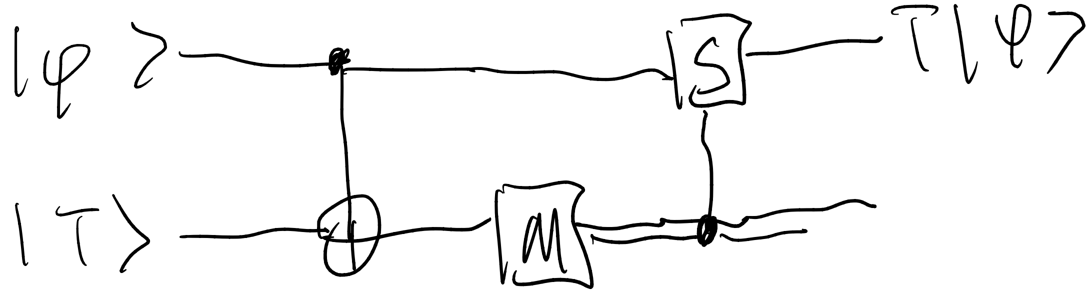
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