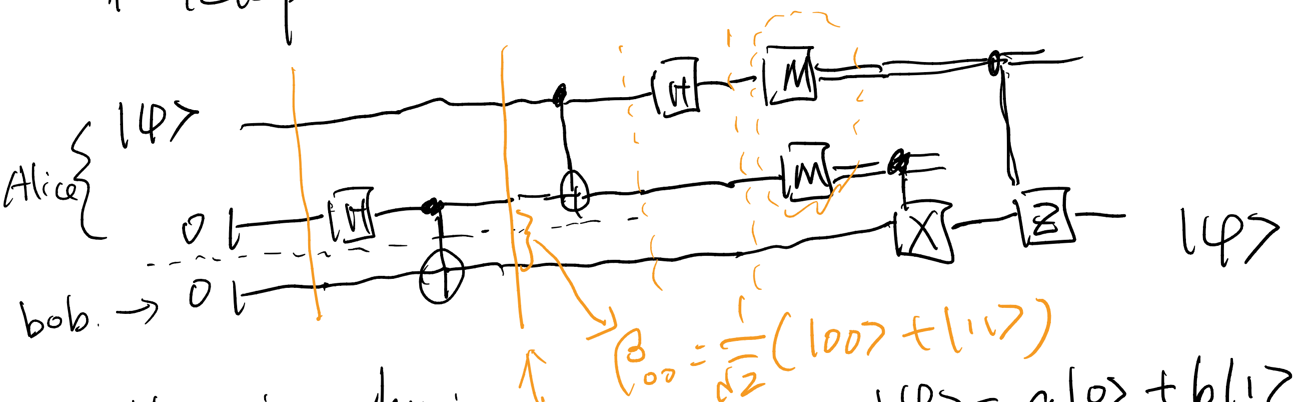
1/23/2025

* Teleportation circuit.



Here is why:

$|\varphi\rangle = a|0\rangle + b|1\rangle$, $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$

$$|\varphi\rangle|00\rangle \longmapsto |\varphi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= (a|0\rangle + b|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)$$

$CNot_{1\to 2} \otimes Id$

$$\longmapsto \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle)$$

$H \otimes Id \otimes Id$

$$\longmapsto \frac{1}{\sqrt{2}}(a|+\rangle \otimes |00\rangle + a|+\rangle \otimes |11\rangle + b|-\rangle \otimes |10\rangle + b|-\rangle \otimes |01\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(a\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |00\rangle + a\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |11\rangle\right.$$

$$\left. + b\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |10\rangle + b\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |01\rangle\right.$$

$$= \frac{1}{2}\big(a|000\rangle + a|100\rangle + a|011\rangle + a|111\rangle$$

$$+ b|010\rangle - b|110\rangle + b|001\rangle - b|101\rangle\big)$$

$$= \frac{1}{2}\big(|00\rangle \otimes (a|0\rangle + b|1\rangle)$$

$$+ |01\rangle \otimes (a|1\rangle + b|0\rangle)$$

$$+ |10\rangle \otimes (a|0\rangle - b|1\rangle)$$

$$+ |11\rangle \otimes (a|1\rangle - b|0\rangle) \big)$$

$M \otimes M \otimes Id$

$\longmapsto$

① $|00\rangle$ $|\varphi\rangle$ is the state of the 3rd qubit.

② $|01\rangle$
bob has to apply a X gate to recover $|\varphi\rangle$.

③ $|10\rangle$
Bob has to apply a Z gate to recover $|\varphi\rangle$.

④ $|11\rangle$, bob has to first apply X, then apply Z to get $|\varphi\rangle$.

* Well-known facts and definitions.

① A gate set S is universal for quantum computing: for any unitary operation U, and for any precision ε, there exists a circuit C constructed from S s.t. $\|C - U\| < \varepsilon$.

* We define the Clifford circuits to be the set of circuits constructed from the gate set $\{H, S, \text{CNot}\}$.

Z gate is Clifford because

$$S^2 = Z.$$

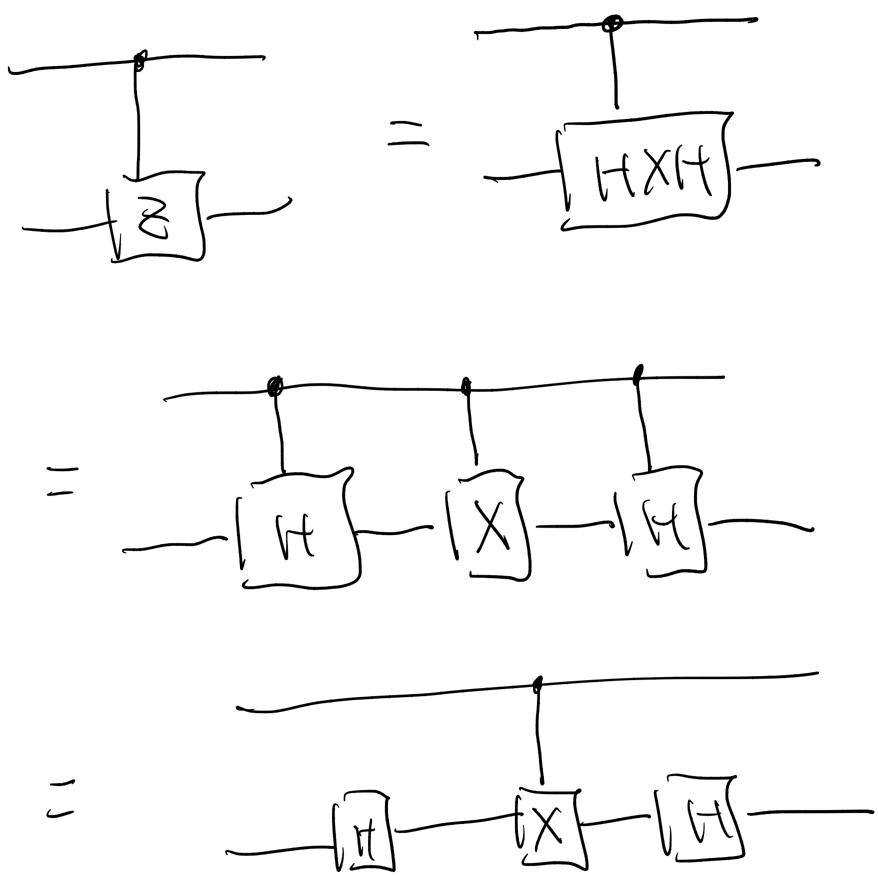
X gate is also Clifford because

$$X = H \circ Z \circ H \quad \text{(leave as an excise.)}$$

$$\Downarrow$$

$$HXH = Z$$

CZ gate is also Clifford.



So CZ is Clifford.

Other important properties:

① Clifford circuits can be efficiently simulated by a classical computer.

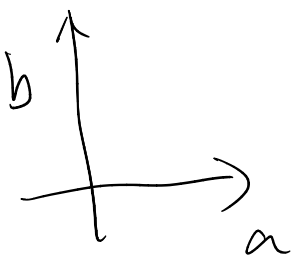
Note: T gate and Toffoli gate are not Clifford.

* Some examples of universal gate set.

① Clifford + T.

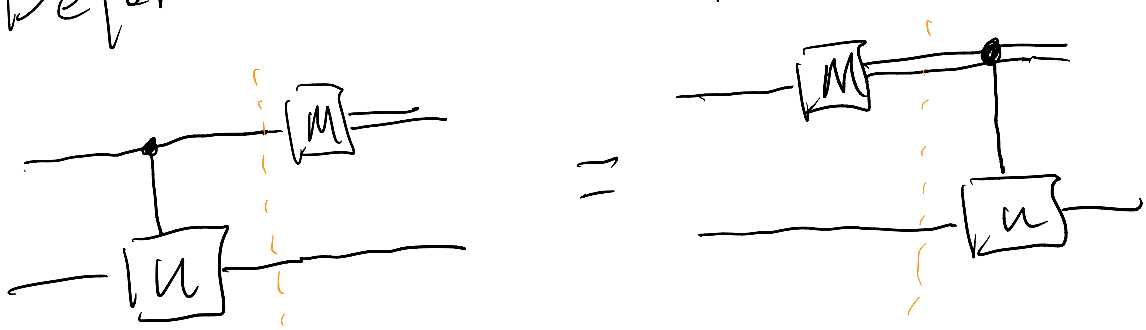
② Clifford + CS (controlled S gate).

③ $\{ \text{Toffoli}, H \}$



$a + bi$

* Defer measurement principle.



$|\varphi\rangle = a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle$.

LHS: $|\varphi\rangle \xmapsto{\text{controlled } U} a_1|00\rangle + a_2|01\rangle + a_3|1\rangle \otimes U|0\rangle$

$+ a_4|1\rangle \otimes U|1\rangle$

$= |0\rangle \otimes (a_1|0\rangle + a_2|1\rangle)$

$+ |1\rangle \otimes (a_3 U|0\rangle + a_4 U|1\rangle)$

$$\xmapsto{M\otimes Id}\begin{cases} ① \ |0\rangle, \text{ then } a_1|0\rangle + a_2|1\rangle \\ ② \ |1\rangle, \text{ then } a_3 U|0\rangle + a_4 U|1\rangle \end{cases}$$

$$|\psi\rangle = a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle$$

$$\text{RHS: } |\psi\rangle \xmapsto{M\otimes Id} \begin{cases} \text{if } 0, \ \underline{a_1|0\rangle + a_2|1\rangle} \\ \\ \text{if } 1, \ a_3|0\rangle + a_4|1\rangle \end{cases}$$

apply U

$$= a_3 U|0\rangle + a_4 U|1\rangle$$