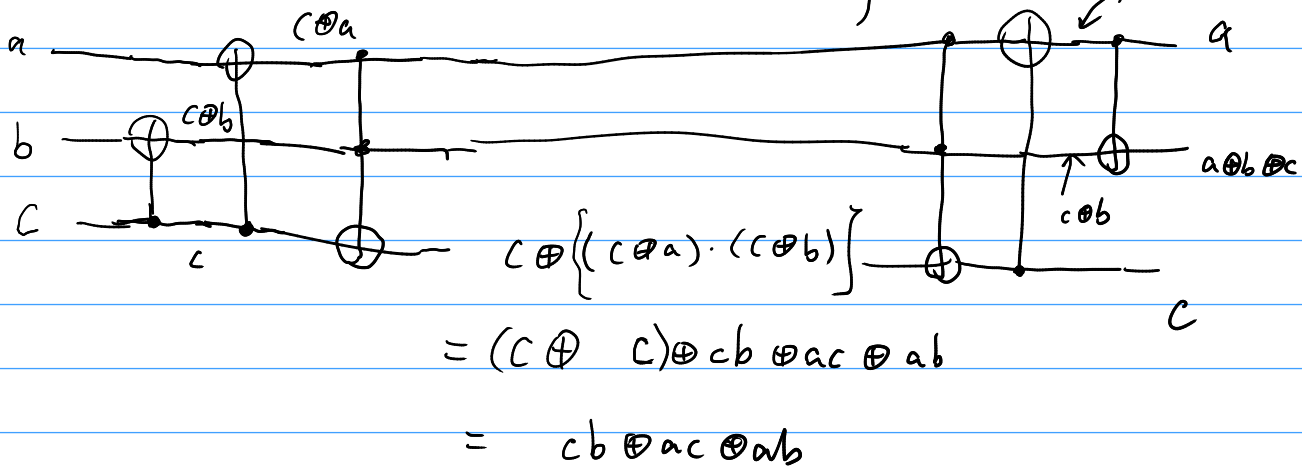


maj - circuit.



* QFT (Quantum Fourier Transform)

$$qft |a\rangle = \frac{1}{\sqrt{2^n}} \sum_k^{2^n-1} e^{2\pi i \frac{a \cdot k}{2^n}} |k\rangle$$

$a \in \{0, \dots, N\}, \quad N = 2^n$

a is a n-bit binary.

e.g.

$$\begin{aligned} |0\rangle &= |000\rangle \\ |1\rangle &= |001\rangle \\ |2\rangle &= |010\rangle \\ &\vdots \\ |7\rangle &= |111\rangle \end{aligned}$$

$$e^{2\pi i \frac{a}{2}} = e^{\pi i \cdot a} = (-1)^a$$

① qft is unitary.

n=1

$$qft |a\rangle = \frac{1}{\sqrt{2}} \sum_{k=0}^1 e^{2\pi i \frac{a \cdot k}{2}} |k\rangle$$

$a \in \{0, 1\}$

$$= \frac{1}{\sqrt{2}} (|0\rangle + (-1)^a |1\rangle)$$

② $qft_1 = H$, hadamard gate.

Thm:

$$\textcircled{3} \text{ qft}_n |a\rangle = \phi_1(a) \otimes \phi_2(a) \dots \otimes \phi_n(a)$$

where $\phi_k(a) = \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i \frac{a}{2^k}} |1\rangle)$

$$n=2$$

$$\text{qft}_2 |a\rangle = \phi_1(a) \otimes \phi_2(a)$$

LHS, by def.

$$\text{qft}_2 |a\rangle = \frac{1}{\sqrt{2}} \sum_{k=0}^1 e^{2\pi i \frac{a \cdot k}{2^2}} |k\rangle$$

$$\text{RHS} = \frac{1}{2} (|0\rangle + e^{2\pi i \frac{a}{2}} |1\rangle) \otimes (|0\rangle + e^{2\pi i \frac{a}{2^2}} |1\rangle)$$

$$= \frac{1}{2} (|00\rangle + e^{2\pi i \frac{a}{2^2}} |01\rangle + e^{2\pi i \frac{a}{2}} |10\rangle$$

$$+ e^{2\pi i (\frac{a}{2^2} + \frac{a}{2})} |11\rangle)$$

def. $\text{qft}_1 |a\rangle = \phi_1(a)$

$$\text{qft}_n |a\rangle = \text{qft}_{n-1} |a\rangle \otimes \phi_n(a)$$

notations: $\exp(x) = e^{2\pi i x}$ $a = a_n a_{n-1} \dots a_1$

binary fraction notation $\frac{a}{2^k} = a_n a_{n-1} \dots a_{k+1} . a_k \dots a_1$

* Note $\exp(a_n \cdots a_{k+1} \cdot a_k \cdots a_1)$

$$= \exp(0 \cdot a_k \cdots a_1)$$

$$\text{So } \phi_k(a) = \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i \frac{a}{2^k}} |1\rangle)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + \exp(0 \cdot a_k \cdots a_1) |1\rangle)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + \exp(0 \cdot a_k + 0 \cdot 0 a_{k-1} + \cdots + 0 \cdot 0 \cdots 0 a_1) |1\rangle)$$

$$|\psi(\theta)\rangle = a|0\rangle + b|1\rangle$$

$$= a|0\rangle + b e^{i\theta} |1\rangle$$

$$\text{Set } \theta = 2\pi \frac{a}{2^k}$$

$$R_k |a\rangle = e^{2\pi i \frac{a}{2^k}} |a\rangle$$

$$a \in \{0, 1\}. \quad = \exp\left(\frac{a}{2^k}\right) |a\rangle$$

$$R_k |1\rangle = \exp\left(\frac{1}{2^k}\right) |1\rangle = \exp\left(0 \cdot 0 \cdots 0 \cdot 1\right)$$

$$P(\theta_2) (P(\theta_1) |a\rangle) = P(\theta_1 + \theta_2) |a\rangle$$

$$CR_k |a, b\rangle = \exp\left(\frac{ab}{2^k}\right) |ab\rangle$$

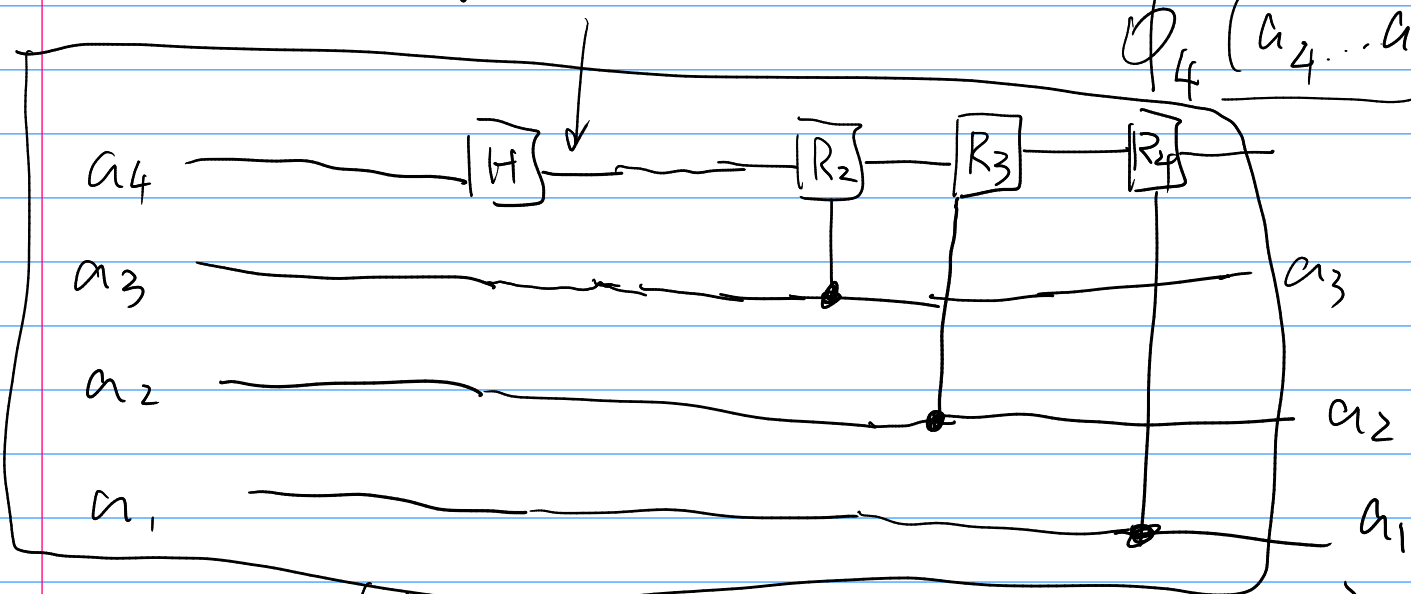
$$a, b \in \{0, 1\}$$

$$\phi_4(a_4 a_3 a_2 a_1) = \frac{1}{\sqrt{2}} (|0\rangle + \exp(0 \cdot a_4 + 0 \cdot 0 a_3 + 0 \cdot 0 0 a_2 + 0 \cdot 0 0 0 a_1) |1\rangle)$$

Ψ_4

$$\frac{1}{\sqrt{2}} (|0\rangle + \exp(0 \cdot a_4) |1\rangle)$$

$\phi_4(a_4 \dots a_1)$



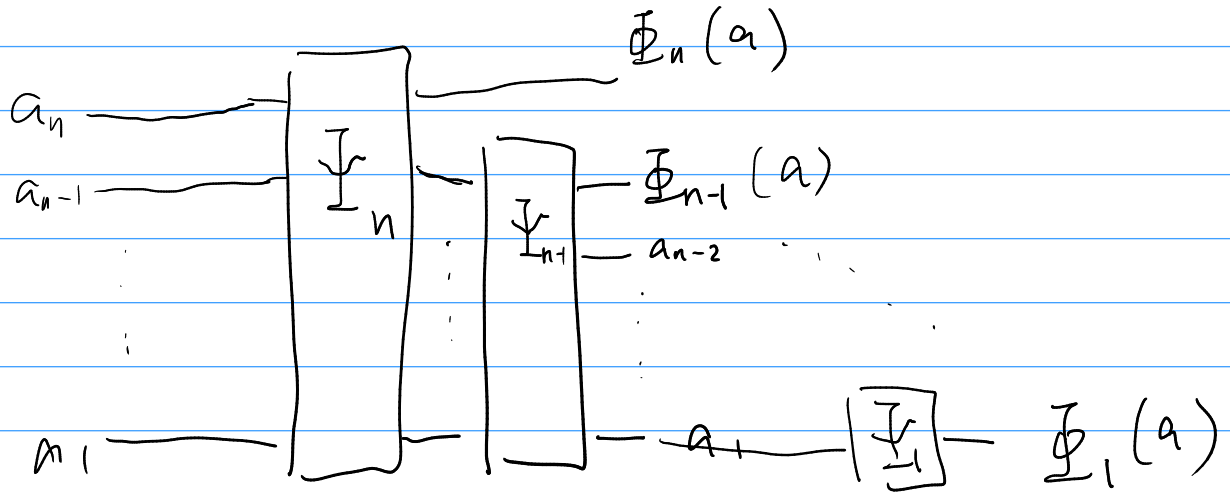
$$CR_2 \left(\frac{1}{\sqrt{2}} |0\rangle \otimes |a_3\rangle + \exp(0 \cdot a_4) |1\rangle \otimes |a_3\rangle \right)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle \otimes |a_3\rangle + \exp(0 \cdot a_4) \cdot \exp(0 \cdot 0 a_3) |1\rangle \otimes |a_3\rangle)$$

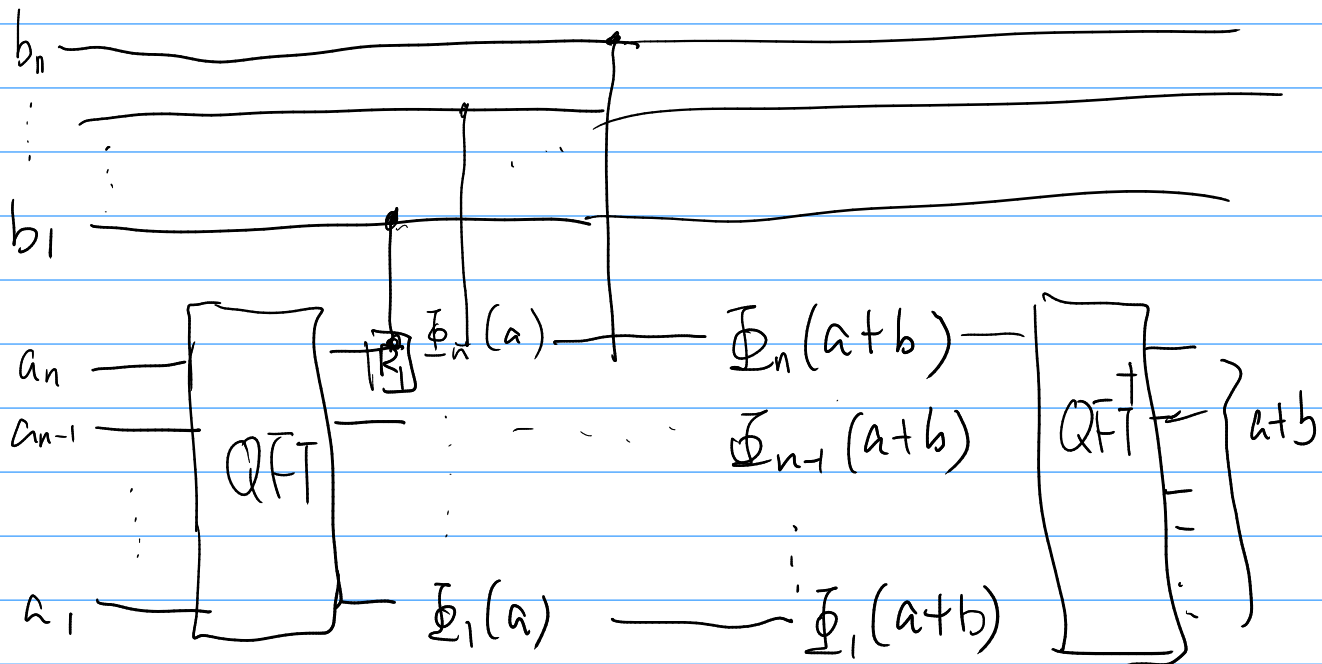
$$|1\rangle \otimes |a_3\rangle$$

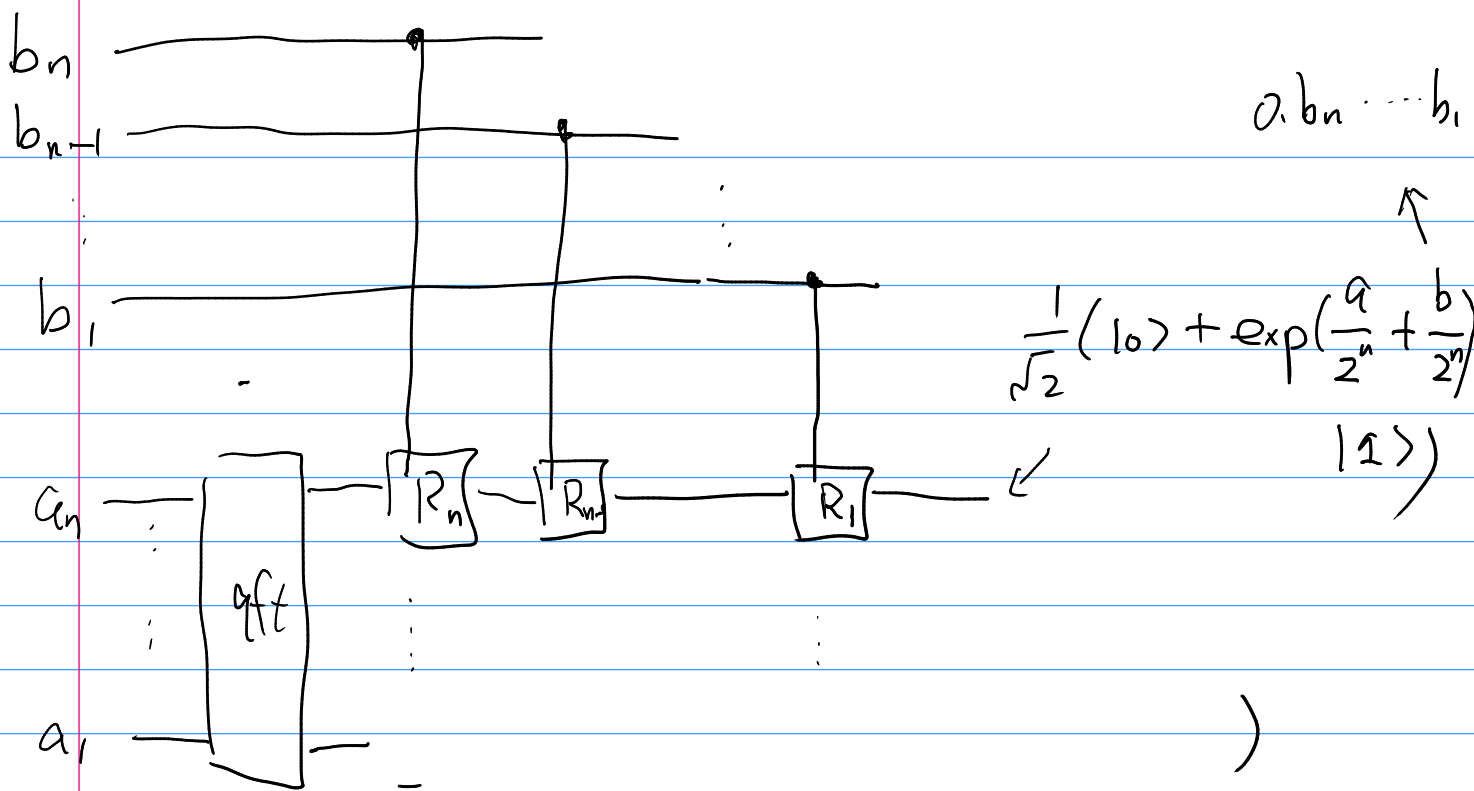
$$= \frac{1}{\sqrt{2}} (|0\rangle + \exp(i\alpha_4 + i\alpha_3) |1\rangle)$$

$$\otimes |a_3\rangle$$



QFT circuit.





$$R_k |a\rangle = \exp\left(\frac{a}{2^k}\right) |a\rangle$$

a_n
 \vdots
 a_1
 $\Phi_n(a) = \frac{1}{\sqrt{2}}(|0\rangle + \exp(\frac{a}{2^n})|1\rangle)$
 $= 0. a_n a_{n+1} \dots a_1$
 $\Phi_1(a) = \frac{1}{\sqrt{2}}(|0\rangle + \exp(\frac{a}{2})|1\rangle)$

