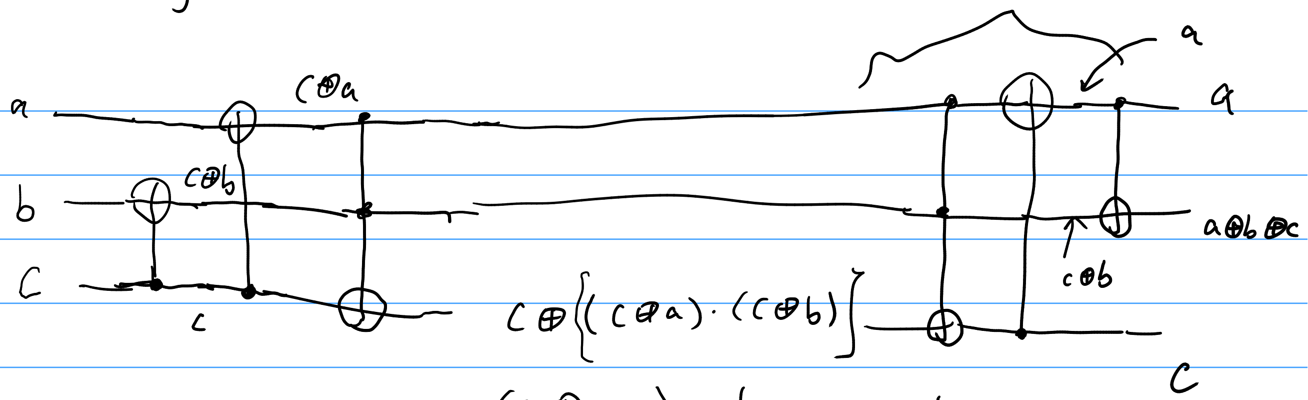
maj - circuit. unmaj.



$$c \oplus \{(c\oplus a)\cdot(c\oplus b)\}$$
$$= (c \oplus c) \oplus cb \oplus ac \oplus ab$$
$$= cb \oplus ac \oplus ab$$

* QFT (Quantum Fourier Transform)

$$qft\,|a\rangle = \frac{1}{\sqrt{2}^n} \sum_{k}^{2^n-1} e^{2\pi i \frac{a\cdot k}{2^n}} |k\rangle$$

$a \in \{0, \dots, N\}$. $N = 2^n$

a is a n-bit binary.

e.g $|0\rangle = |000\rangle$
$|1\rangle = |001\rangle$
$|2\rangle = |010\rangle$
⋮
$|7\rangle = |111\rangle$

$$e^{2\pi i \frac{a}{2}} = e^{\pi i \cdot a} = (-1)^a$$

① qft is unitary.

$n=1$

$$qft\,|a\rangle = \frac{1}{\sqrt{2}} \sum_{k=0}^{1} e^{2\pi i \frac{a\cdot k}{2}} |k\rangle$$

$a \in \{0, 1\}$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^a |1\rangle\right)$$

② $qft_1 = H$, hadamard gate.

Thm:

③ $$qft_n |a\rangle = \phi_1(a) \otimes \phi_2(a) \cdots \otimes \phi_n(a)$$

where $$\phi_k(a) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \frac{a}{2^k}} |1\rangle\right)$$

$$\boxed{\begin{array}{l} n=2 \\ qft_2 |a\rangle = \phi_1(a) \otimes \phi_2(a) \end{array}}$$

LHS, by def.

$$qft_2 |a\rangle = \frac{1}{\sqrt{2}^2} \sum_{k=0}^{3} e^{2\pi i \frac{a \cdot k}{2^2}} |k\rangle$$

$$RHS = \frac{1}{2}\left(|0\rangle + e^{2\pi i \frac{a}{2}} |1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i \frac{a}{2^2}} |1\rangle\right)$$

$$= \frac{1}{2}\Big(|00\rangle + e^{2\pi i \frac{a}{2^2}} |01\rangle + e^{2\pi i \frac{a}{2}} |10\rangle$$

$$+ e^{2\pi i \left(\frac{a}{2^2} + \frac{a}{2}\right)} |11\rangle$$

def. $$qft_1 |a\rangle = \phi_1(a)$$

$$qft_n |a\rangle = qft_{n-1} |a\rangle \otimes \phi_n(a)$$

notations: $\exp(\alpha) = e^{2\pi i \alpha}$ $\qquad a = a_n a_{n-1} \cdots a_1$

binary fraction notation $\frac{a}{2^k} = a_n a_{n-1} \cdots a_{k+1} . a_k \cdots a_1$, $k \geq 1$

* Note $\exp(a_n \cdots a_{k+1} . a_k \cdots a_1)$

$$= \exp(0.a_k \cdots a_1)$$

So $\phi_k(a) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \frac{a}{2^k}} |1\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + \exp(0.a_k \cdots a_1)|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + \exp(0.a_k + 0.0a_{k-1} + \cdots\right.$$
$$\left. 0.\underbrace{0 \cdots 0}a_1)|1\rangle\right)$$

$P(\theta)\left(a|0\rangle + b|1\rangle\right)$

$= a|0\rangle + b e^{i\theta} |1\rangle$

set $\theta = 2\pi \frac{a}{2^k}$?

$R_k |a\rangle = e^{2\pi i \frac{a}{2^k}} |a\rangle$

$a \in \{0,1\}$. $= \exp\left(\frac{a}{2^k}\right)|a\rangle$

$R_k|1\rangle = \exp\left(\frac{1}{2^k}\right)|1\rangle = \exp(0.\underbrace{0 \cdots 1}_{k})|1\rangle$

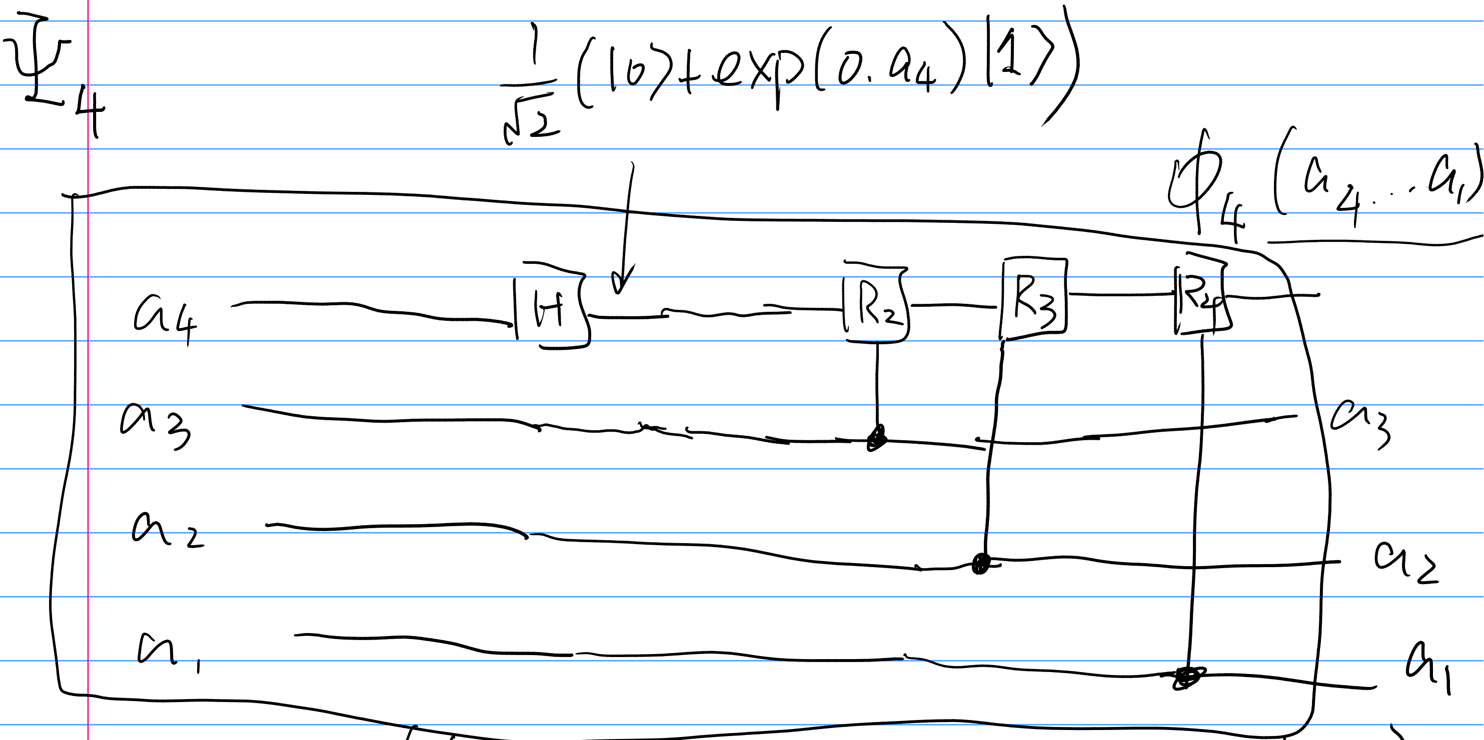
$$P(\theta_2)\big(P(\theta_1)\,|a\rangle\big) = P(\theta_1+\theta_2)\,|a\rangle$$

$$CR_k\,|a,b\rangle = \exp\left(\frac{ab}{2^k}\right)|ab\rangle$$

$a, b \in \{0, 1\}$

$$\phi_4(a_4a_3a_2a_1) = \frac{1}{\sqrt{2}}\big(|0\rangle + \exp(0.a_4 + 0.0a_3 + 0.00a_2 + 0.000a_1)\,|1\rangle\big)$$

Ψ_4

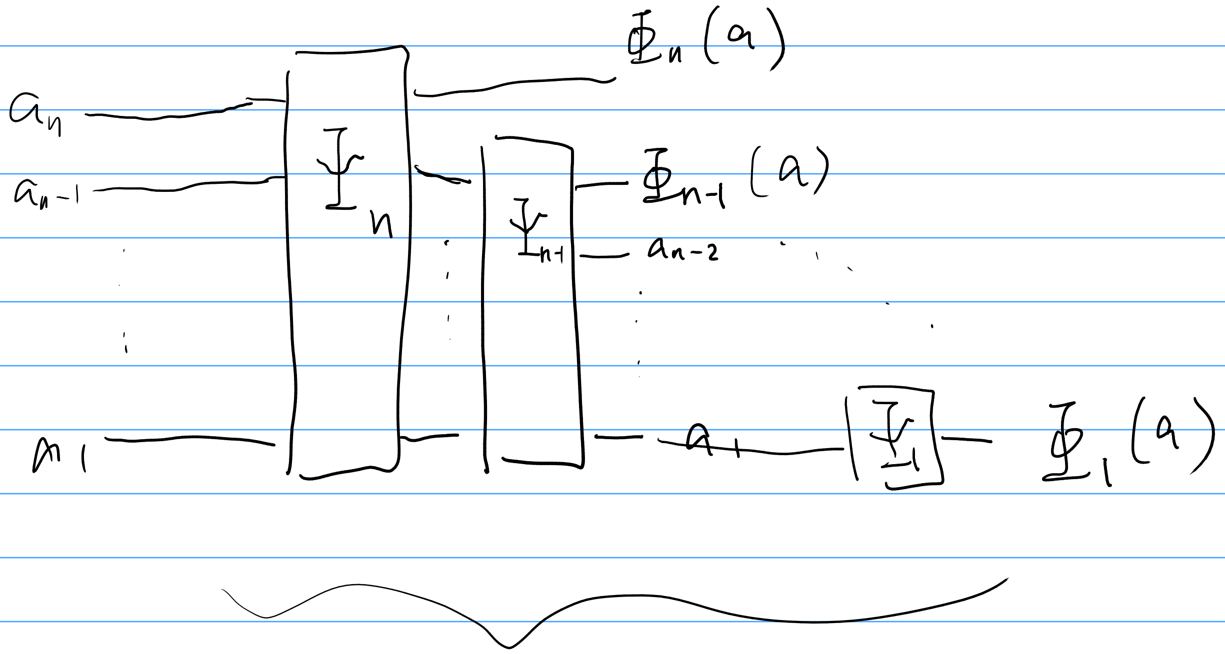


$$CR_2\left(\frac{1}{\sqrt{2}}|0\rangle\otimes|a_3\rangle + \exp(0.a_4)\,|1\rangle\otimes|a_3\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\big(|0\rangle\otimes|a_3\rangle + \exp(0.a_4)\cdot\exp(0.0a_3)\,|1\rangle\otimes|a_3\rangle\big)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + \exp(0.a_4 + 0.0a_3)|1\rangle\right)$$

$$\otimes |a_3\rangle$$



QFT circuit.