



1/14/2025

complex numbers

\* Qubit.

$a|0\rangle + b|1\rangle$ , where  $a, b \in \mathbb{C}$   
"ket notation".  
s.t.  $|a|^2 + |b|^2 = 1$ .

example of a qubit

\*  $|0\rangle$

$$* \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle = |+\rangle$$

$$* \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle = |-\rangle$$

$$* |T\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\frac{\pi}{4}} |1\rangle)$$

$$e^{i\frac{\pi}{4}} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$$

$$|e^{i\frac{\pi}{4}}|^2 = \cos^2 \frac{\pi}{4} + \sin^2 \frac{\pi}{4} = 1$$

\* Multi-qubits.

2-qubit state

$$a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle$$

$$a_1, \dots, a_4 \in \mathbb{C}, |a_1|^2 + |a_2|^2 + \dots + |a_4|^2 = 1.$$

can be generalized to  $n$ -qubit state.

$$* \text{ Qubit} = \{ a|0\rangle + b|1\rangle \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \}$$

$$\text{Qubit} \otimes \text{Qubit} = \{ a_1|00\rangle + \dots + a_4|11\rangle \mid a_1, \dots, a_4 \in \mathbb{C}, |a_1|^2 + \dots + |a_4|^2 = 1 \}$$

$$\text{Qubit}^{\otimes 3}$$

\* Tensor product of qubits.

$$(a_1|0\rangle + a_2|1\rangle) \otimes (b_1|0\rangle + b_2|1\rangle)$$

$$= a_1 b_1 \underbrace{|0\rangle \otimes |0\rangle} + a_1 b_2 |0\rangle \otimes |1\rangle + a_2 b_1 |1\rangle \otimes |0\rangle + a_2 b_2 |1\rangle \otimes |1\rangle$$

$$= a_1 b_1 |00\rangle + a_1 b_2 |01\rangle + a_2 b_1 |10\rangle + a_2 b_2 |11\rangle$$

\* Can we obtain any 2-qubit state by tensoring 2 single qubit states?

Surprisingly, no.

$$\text{bell}_{00} = \beta_{00} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

Claim:

$$\nexists \phi_1, \phi_2 \text{ st. } \beta_{00} = \phi_1 \otimes \phi_2, \text{ where } \phi_1, \phi_2 \in \text{Qubit}$$

\* Operations qubit states.

\*\* Unitary operations.

Suppose  $U : \text{Qubit}^{\otimes n} \rightarrow \text{Qubit}^{\otimes n}$  is a unitary operation.

Then  $U$  has the following properties:

①  $U$  is linear.

$$U(a_1|0\rangle + a_2|1\rangle) = a_1 U|0\rangle + a_2 U|1\rangle$$

②  $\exists$   $G$  : Qubit $^{\otimes n} \rightarrow$  Qubit $^{\otimes n}$  s.t.  
unitary operation

$$U \circ G = G \circ U = I$$

we often write  $U^\dagger$  for  $G$ .

\* Examples of Unitaries.

\* Pauli operations/gates.

$$\{X, Z, Y, I\}$$

$$X : \text{Qubit} \rightarrow \text{Qubit}$$

$$\begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{cases}$$

$$\begin{aligned} X(a|0\rangle + b|1\rangle) &= a(X|0\rangle) + b(X|1\rangle) \\ &= a|1\rangle + b|0\rangle \end{aligned}$$

$$X^\dagger = X$$

\* Z - gate

Z: Qubit  $\rightarrow$  Qubit

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle.$$

$$Z^\dagger = Z \leftarrow \text{Hermitian}$$

\*  $I|0\rangle = |0\rangle$

$I|1\rangle = |1\rangle.$

\*  $Y|0\rangle = i|1\rangle$

$Y|1\rangle = -i|0\rangle$

$i^2 = -1$ , so  $Y^\dagger = Y.$

$$Z \circ X = iY$$

$$(Z \circ X) |0\rangle = -|1\rangle$$

$$iY |0\rangle = i(i|1\rangle) = -|1\rangle$$

$$(Z \circ X) |1\rangle = Z |0\rangle = |0\rangle$$

$$(iY) |1\rangle = i(-i) |0\rangle = |0\rangle.$$

\* Couple of things about Pauli gates

① They are hermitian,  
i.e.  $P^\dagger = P$ ,  $P \in \{X, Y, Z, I\}$

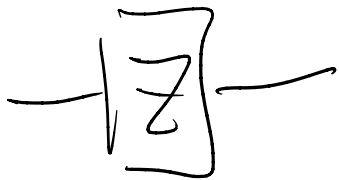
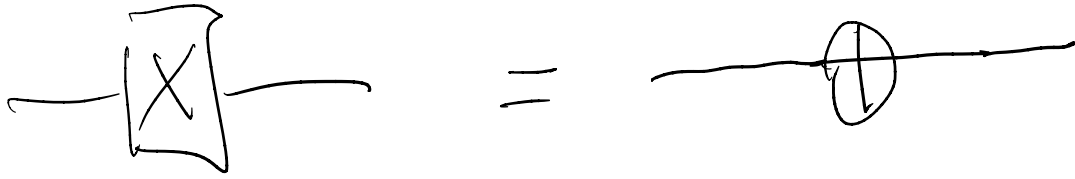
②  $P_1, P_2 \in \{X, Y, Z\}$ , and

$$P_1 \neq P_2, \quad P_1 \circ P_2 = -P_2 \circ P_1$$

"anti-commutation.

$$X \circ Z = -Z \circ X.$$

\* Circuit notation for Pauli gates.



\* Hadamard gate.

$H: \text{Qubit} \rightarrow \text{Qubit}$ .

$$H |0\rangle = |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$H |1\rangle = |-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

$$H^\dagger = H$$



$$(H \circ H) |0\rangle = H \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} H |0\rangle + \frac{1}{\sqrt{2}} H |1\rangle$$

$$= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right)$$

$$+ \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$$

$$= |0\rangle.$$

$$(H \circ H) |1\rangle = |1\rangle.$$

\* Phase gate  $\theta \in [0, 2\pi]$

$$P(\theta) |0\rangle = |0\rangle$$

$$P(\theta) |1\rangle = e^{i\theta} |1\rangle$$

$$P(\theta)^T = P(-\theta)$$

\*  $Z = P(\pi)$

$$e^{i\theta}$$

