1/14/2025

complex numbers

* Qubit.

$a|0\rangle + b|1\rangle$, where $a, b \in \mathbb{C}$

s.t. $|a|^2 + |b|^2 = 1$.

"ket notation".

example of a qubit

* $|0\rangle$

* $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$

* $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle$

* $|T\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle)$

$$e^{i\frac{\pi}{4}} = \underline{\cos\frac{\pi}{4}} + i\sin\frac{\pi}{4}$$

$$|e^{i\frac{\pi}{4}}|^2 = \cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4} = 1$$

* Multi-qubits.

2-qubit state

$$a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle$$

$$a_1, \cdots a_4 \in \mathbb{C}, \quad |a_1|^2 + |a_2|^2 + \cdots + |a_4|^2 = 1.$$

⋮

can be generalized to n-qubit state.

$$\text{* Qubit} = \{ a|0\rangle + b|1\rangle \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \}.$$

$$\text{Qubit} \otimes \text{Qubit} = \{ a_1|00\rangle + \cdots + a_4|11\rangle \mid a_1, \cdots a_4 \in \mathbb{C}, |a_1|^2 + \cdots |a_4|^2 = 1 \}.$$

$$\text{Qubit}^{\otimes 3}$$

* Tensor product of qubits.

$$(a_1|0\rangle + a_2|1\rangle) \otimes (b_1|0\rangle + b_2|1\rangle)$$

$$= a_1 b_1 |0\rangle \otimes |0\rangle + a_1 b_1 |0\rangle \otimes |1\rangle + a_2 b_1 |1\rangle \otimes |0\rangle + a_2 b_2 |1\rangle \otimes |1\rangle$$

$= a_1 \cdot b_1 |00\rangle + a_1 b_2 |01\rangle + a_2 b_1 |10\rangle + a_2 b_2 |11\rangle$

* Can we obtain any 2-qubit state by tensoring 2 single qubit states?

surprisingly, no.

$\text{bell}_{00} = \beta_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Claim:

$\nexists \phi_1, \phi_2$ s.t. $\beta_{00} = \phi_1 \otimes \phi_2$, where $\phi_1, \phi_2 \in$ Qubit

* Operations qubit states.

** Unitary operations.

Suppose U : $\text{Qubit}^{\otimes n} \to \text{Qubit}^{\otimes n}$ is a unitary operation.

Then U has the following properties:

① $\mathcal{U}$ is linear.

$$\mathcal{U}(a_1|0\rangle + a_2|1\rangle) = a_1\mathcal{U}|0\rangle + a_2\mathcal{U}|1\rangle.$$

② $\exists\, G : \text{Qubit}^{\otimes n} \to \text{Qubit}^{\otimes n}$ s.t.

unitary operation

$$\mathcal{U} \circ G = G \circ \mathcal{U} = I$$

we often write $\mathcal{U}^\dagger$ for G.

* Examples of Unitaries.

* Pauli operations/gates.

$\{X, Z, Y, I\}$.

$X : \text{Qubit} \to \text{Qubit}$

$$\begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{cases}$$

$$\begin{aligned} X(a|0\rangle + b|1\rangle) &= a(X|0\rangle) + b(X|1\rangle) \\ &= a|1\rangle + b|0\rangle \end{aligned}$$

$X^\dagger = X$

* Z - gate

Z : Qubit $\to$ Qubit

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$.

$Z^\dagger = Z$ $\leftarrow$ Hermitian

* $I|0\rangle = |0\rangle$

$I|1\rangle = |1\rangle$.

* $Y|0\rangle = i|1\rangle$

$Y|1\rangle = -i|0\rangle$

$i^2 = -1$, so $Y^\dagger = Y$.

$Z \circ X = iY$

$(Z \circ X)|0\rangle = -|1\rangle$

$iY|0\rangle = i(i|1\rangle) = -|1\rangle$

$(Z \circ X)|1\rangle = Z|0\rangle = |0\rangle$

$(i \cdot Y)|1\rangle = i \cdot (-i)|0\rangle = |0\rangle$.

* Couple of things about Pauli gates.

① They are hermitian,

i.e. $P^\dagger = P$, $P \in \{X, Y, Z, \}$

② $P_1, P_2 \in \{X, Y, Z\}$, and

$P_1 \neq P_2$, $P_1 \circ P_2 = -P_2 \circ P_1$

"anti-commutation.

$X \circ Z = -Z \circ X$.

* Circuit notation for Pauli gates.

$$-\boxed{X}- \quad = \quad -\oplus-$$

$$-\boxed{Z}-$$

$$-\boxed{Y}-$$

* Hadamard gate.

H : Qubit $\rightarrow$ Qubit.

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H^{\dagger} = H$$

$$(H \circ H)\,|0\rangle = H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}} H|0\rangle + \frac{1}{\sqrt{2}} H|1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \cancel{\frac{1}{\sqrt{2}}|1\rangle}\right.$$

$$\left. + \frac{1}{\sqrt{2}}|0\rangle - \cancel{\frac{1}{\sqrt{2}}|1\rangle}\right)$$

$$= |0\rangle.$$

$$(H \circ H)\,|1\rangle = |1\rangle.$$

* Phase gate $\theta \in [0, 2\pi]$

$$P(\theta)\,|0\rangle = |0\rangle$$

$$P(\theta)\,|1\rangle = e^{i\theta}\,|1\rangle$$

$$P(\theta)^{\dagger} = P(-\theta)$$

$$\ast \quad Z = P(\pi)$$

$$e^{i\theta}$$