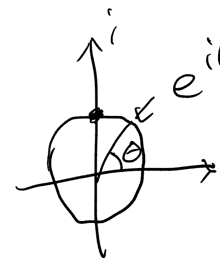
1/16/2025

$P(\theta)$

* T gate. $T = P(\frac{\pi}{4})$



$$T|0\rangle = |0\rangle$$
$$T|1\rangle = e^{i\frac{\pi}{4}}|1\rangle.$$

* S-gate. $S = P(\frac{\pi}{2})$ Note that $e^{i\frac{\pi}{2}} = i$

$$S|0\rangle = |0\rangle$$
$$S|1\rangle = i|1\rangle$$

* Note that $T \circ T = T^2 = S$

$$\boxed{S^2 = Z \qquad Z^2 = I.}$$

$$T^\dagger = P(-\tfrac{\pi}{4}) = T^7$$
$$T^7 \cdot T = T^8 = I$$
$$S^\dagger = S^3$$

* Multi qubit gate.

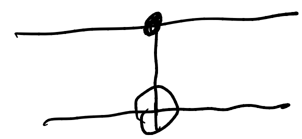
2-qubit gate.

* control-not gate (Notation: CNOT, CX)

Notation: $Q^2 := \text{Qubit} \otimes \text{Qubit}.$

cnot: $Q^2 \to Q^2$

circuit notation.



$\text{cnot}\,|00\rangle = |00\rangle$

$\text{cnot}\,|01\rangle = |01\rangle$

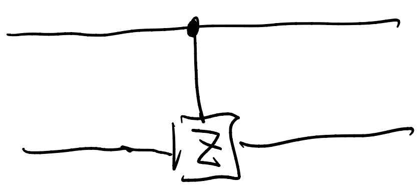
$\text{cnot}\,|10\rangle = |11\rangle$

$\text{cnot}\,|11\rangle = |10\rangle$

control bit — target bit.

* Control Z gate (notation CZ)

circuit notation:



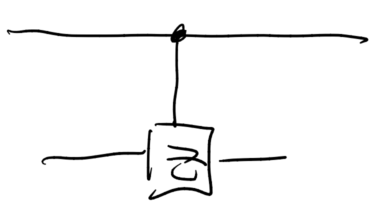
$CZ\,|00\rangle = |00\rangle$

$CZ\,|01\rangle = |01\rangle$

$CZ\,|10\rangle = |10\rangle$

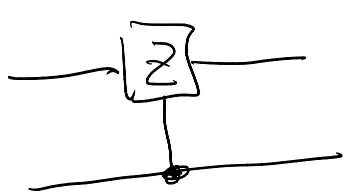
$CZ\,|11\rangle = -|11\rangle$

Note:



=

CZ'



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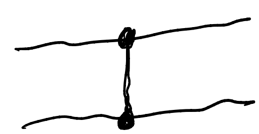
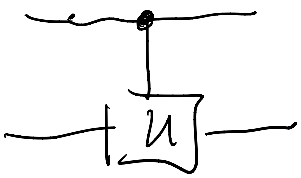
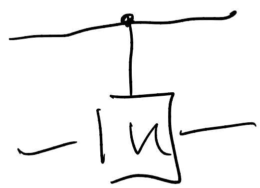


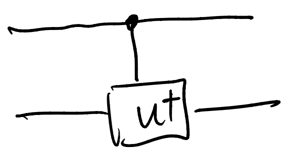
* In general if U is a unitary gate

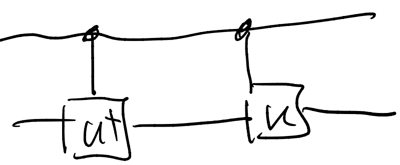


controlled-U gate.

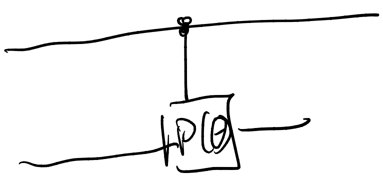
$$U : Q^n \to Q^n$$

CU : 

$CU^\dagger$: 

$CU \circ CU^\dagger$: 

* Remark:



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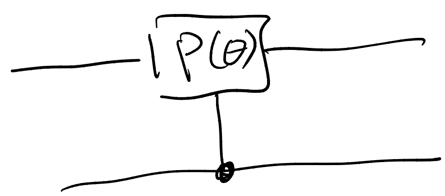

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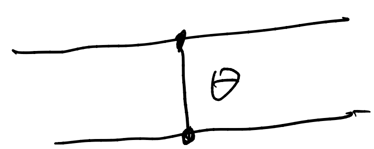
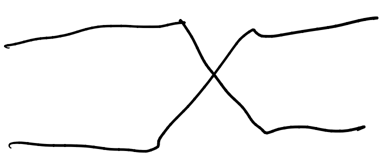


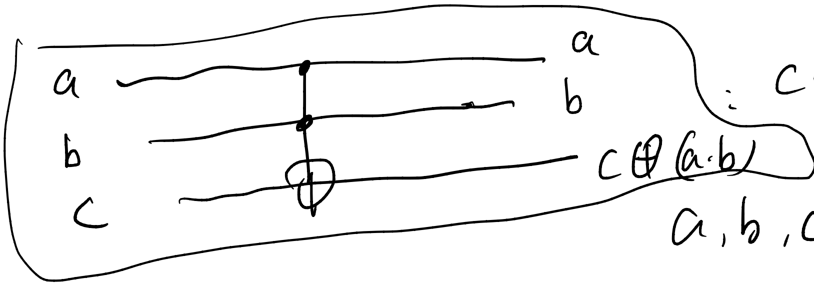
* 

swap gate.

$$\text{swap}\,|a,b\rangle = |b,a\rangle$$

* 3-qubit gate.

Toffoli gate (doubly controlled not gate)



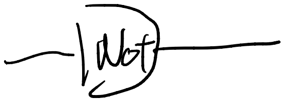
$: \; CCX\,|a, b, c\rangle = |a, b, c \oplus a{\cdot}b\rangle$

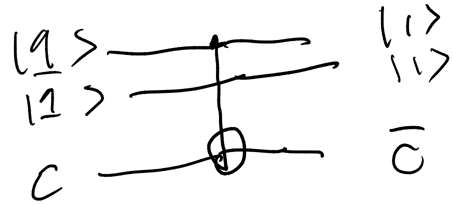
$a, b, c \in \{0, 1\}$

$c \oplus 1 = \bar{c}$

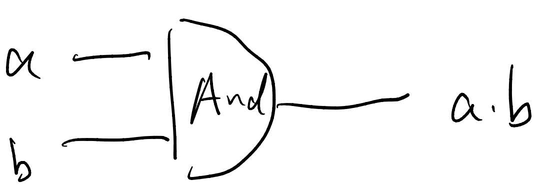
* Classical boolean Circuit.

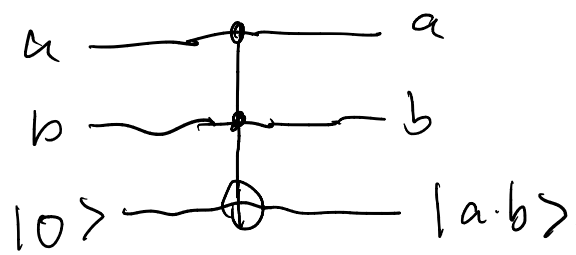
NOT - gate. using Toffoli gate.





And - gate using Toffoli.

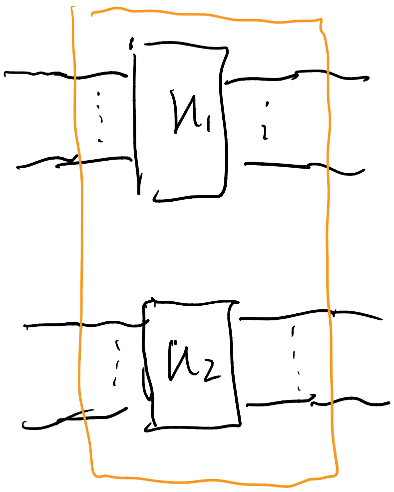




* constructing multi-qubit gate (circuit) by tensoring.

$$U_1 : Q^n \to Q^n$$
$$U_2 : Q^m \to Q^m$$
$$\overline{U_1 \otimes U_2 : Q^{n+m} \to Q^{n+m}}$$

circuit notation



$u_1 \otimes u_2$

e.g. $(H \otimes H)|00\rangle$

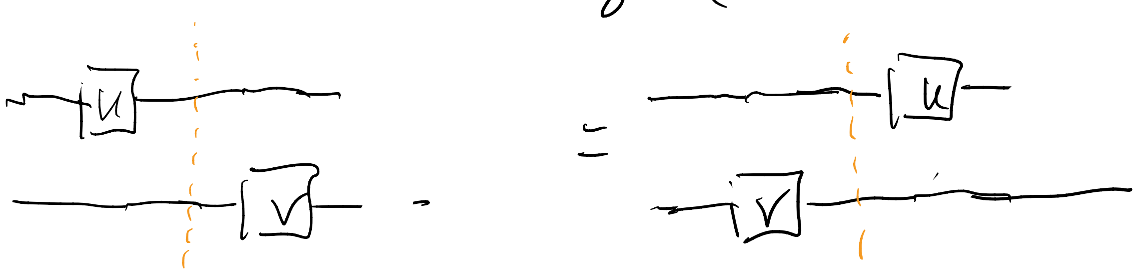
$= (H \otimes H)(|0\rangle \otimes |0\rangle)$

$= H|0\rangle \otimes H|0\rangle$

$= |+\rangle \otimes |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

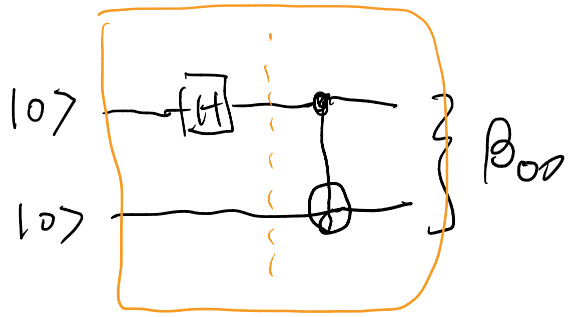
$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

* We can construct a quantum circuit from a finite set of gates S, via function composition (horizontal composition) and tensoring (vertical composition).



In "algebraic notation" we have

$$(Id \otimes V) \circ (U \otimes Id) = (U \otimes Id) \circ (Id \otimes V)$$

* 

Note $H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Bell state $\beta_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\beta_{01}, \beta_{10}, \beta_{11}$