Quantum Programming Languages
CSCE 790 Section 008 Homework 3

Recall that we have the following typing rules for Linear typed Lambda Calculus.

$$\frac{}{\Phi, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \multimap B}$$

$$\frac{\Phi, \Gamma_1 \vdash M : A \multimap B \quad \Phi, \Gamma_2 \vdash N : A}{\Phi, \Gamma_1, \Gamma_2 \vdash MN : B} \qquad \frac{\Phi, \Gamma_1 \vdash M : A \quad \Phi, \Gamma_2 \vdash N : B}{\Phi, \Gamma_1, \Gamma_2 \vdash (M, N) : A \otimes B}$$

$$\frac{\Phi, \Gamma_1 \vdash M : A \otimes B \quad \Phi, \Gamma_2, x : A, y : B \vdash N : C}{\Phi, \Gamma_1, \Gamma_2 \vdash \textbf{let } (x, y) = M \textbf{ in } N : C} \qquad \frac{\Phi \vdash M : A}{\Phi \vdash \textbf{lift} M : !A}$$

$$\frac{\Gamma \vdash M : !A}{\Gamma \vdash \textbf{force} M : A} \qquad \frac{}{\Phi \vdash () : \textbf{Unit}}$$

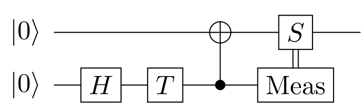
$$\frac{}{\Phi \vdash \textbf{True} : \textbf{Bool}} \qquad \frac{}{\Phi \vdash \textbf{False} : \textbf{Bool}}$$

$$\frac{\Phi, \Gamma_1 \vdash M : \textbf{Bool} \quad \Phi, \Gamma_2 \vdash N_1 : C \quad \Phi, \Gamma_2 \vdash N_2 : C}{\Phi, \Gamma_1, \Gamma_2 \vdash \textbf{if } M \textbf{ then } N_1 \textbf{ else } N_2 : C}$$

Note that parameter types are $P ::= \textbf{Unit} \mid \textbf{Bool} \mid P \otimes P' \mid !A$ and Φ is a parameter context, i.e., $\Phi = x_1 : P_1, ..., x_n : P_n$ (when $n = 0$, we have an empty parameter context).

1. Consider the following closed lambda terms (i.e., they do not contain free variables). Determine if the following typing judgment are valid. If a typing judgment is valid, give a typing derivation using the typing rules specified above. If not, explain why it is not valid.
   (a) (2 points) $\vdash \lambda x.() : \textbf{Qubit} \multimap \textbf{Unit}$
   (b) (2 points) $\vdash \lambda x.() : \textbf{Bool} \multimap \textbf{Unit}$
   (c) (2 points) $\vdash \lambda x.\textbf{lift}\, x : \textbf{Qubit} \multimap !\textbf{Qubit}$.
   (d) (2 points) $\vdash \lambda z.\textbf{let}\, (x, y) = z \textbf{ in } y : \textbf{Qubit} \otimes \textbf{Qubit} \multimap \textbf{Qubit}$.
   (e) (2 points) $\vdash \lambda z.\textbf{let}\, (x, y) = z \textbf{ in } y : \textbf{Bool} \otimes \textbf{Qubit} \multimap \textbf{Qubit}$.
   (f) (2 points) $\vdash \lambda x.\lambda z.\textbf{if}\, z \,\textbf{then}\, x \,\textbf{else}\, x : \textbf{Qubit} \multimap \textbf{Bool} \multimap \textbf{Qubit}$.
   (g) (2 points) $\vdash \lambda x.\lambda y.\lambda z.\textbf{if}\, z \,\textbf{then}\, y \,\textbf{else}\, z : \textbf{Qubit} \multimap \textbf{Qubit} \multimap \textbf{Bool} \multimap \textbf{Qubit}$.

2. (4 points) Write a term for the following quantum circuit, also please provide the type of your term.



You may assume the following gates and their types.

- $\Phi \vdash H : !(\mathbf{Qubit} \multimap \mathbf{Qubit})$
- $\Phi \vdash T : !(\mathbf{Qubit} \multimap \mathbf{Qubit})$
- $\Phi \vdash S : !(\mathbf{Qubit} \multimap \mathbf{Qubit})$
- $\Phi \vdash \text{Meas} : !(\mathbf{Qubit} \multimap \mathbf{Bool})$
- $\Phi \vdash CX : !(\mathbf{Qubit} \otimes \mathbf{Qubit} \multimap \mathbf{Qubit} \otimes \mathbf{Qubit})$
- $\Phi \vdash \mathsf{init0} : !(\mathbf{Unit} \multimap \mathbf{Qubit})$

3. (8 points) Suppose the term you write for the above circuit is N, write down the call-by-value evaluation trace for the configuration $(1, [], N)$. Please refer to the essential evaluation rules below.

$$\frac{\text{fresh } \ell \quad L' = L, \ell}{(\phi, L, (\mathbf{force}\ \mathsf{init0})\ ()) \rightsquigarrow (\phi \otimes |0\rangle, L', \ell)}$$

$$\frac{\phi' = [\![U]\!]_{\ell_i \ldots \ell_j}(\phi)}{(\phi, L, (\mathbf{force}\ U)\ (\ell_i, ..., \ell_j)) \rightsquigarrow (\phi', L, (\ell_i, ..., \ell_j))}$$

$$\frac{\phi = a\psi + b\gamma \quad \psi_{\ell_i} = |0\rangle \quad \gamma_{\ell_i} = |1\rangle}{(\phi, L, (\mathbf{force}\ \text{Meas})\ \ell_i) \overset{|a|^2}{\rightsquigarrow} (\psi/|0\rangle_{\ell_i}, L/\ell_i, \mathbf{False})}$$

$$\frac{\phi = a\psi + b\gamma \quad \psi_{\ell_i} = |0\rangle \quad \gamma_{\ell_i} = |1\rangle}{(\phi, L, (\mathbf{force}\ \text{Meas})\ \ell_i) \overset{|b|^2}{\rightsquigarrow} (\gamma/|0\rangle_{\ell_i}, L/\ell_i, \mathbf{True})}$$

$$\frac{}{(\phi, L, (\lambda x.M)\ V) \rightsquigarrow (\phi, L, [V/x]M)}$$

$$\frac{}{(\phi, L, \mathbf{if}\ \ \mathbf{True\ then}\ \ M\ \ \mathbf{else}\ \ N) \rightsquigarrow (\phi, L, M)}$$

$$\frac{}{(\phi, L, \mathbf{if}\ \ \mathbf{False\ then}\ \ M\ \ \mathbf{else}\ \ N) \rightsquigarrow (\phi, L, N)}$$

$$\frac{}{(\phi, L, \mathbf{force}\ (\mathbf{lift}\ M)) \rightsquigarrow (\phi, L, M)}$$

$$\frac{}{(\phi, L, \mathbf{let}\ (x, y) = (V_1, V_2)\ \mathbf{in}\ M) \rightsquigarrow (\phi, L, [V_1/x, V_2/y]M)}$$