Quantum Programming Languages
CSCE 790 Section 008 Homework 2

Recall that we have the following typing rules for Simply Typed Lambda Calculus with sums and products.

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash (M, N) : A \times B} \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathbf{fst}(M) : A} \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathbf{snd}(M) : B}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathbf{left}(M) : A + B} \qquad \frac{\Gamma \vdash M : B}{\Gamma \vdash \mathbf{right}(M) : A + B} \qquad \frac{}{\Gamma \vdash () : \mathbf{Unit}}$$

$$\frac{\Gamma \vdash M : \bot}{\Gamma \vdash \mathbf{abort}(M) : A} \qquad \frac{\Gamma \vdash M : A + B \quad \Gamma, x : A \vdash N_1 : C \quad \Gamma, y : B \vdash N_2 : C}{\Gamma \vdash \mathbf{case}(M)\mathbf{of}\{\mathbf{left}(x) \to N_1; \mathbf{right}(y) \to N_2\} : C}$$

1. Consider the following closed lambda terms (i.e., they do not contain free variables). Determine if they are typable under the empty typing context. If a term is typable, give it a type and its typing derivation using the typing rules specified above. If not, explain why it is not typable. For example, the closed lambda term $\lambda x.x$ is typable with a type like $A \to A$, and here is its typing derivation.

$$\frac{\dfrac{}{x : A \vdash x : A}}{\vdash \lambda x.x : A \to A}$$

   (a) (2 points) $\lambda x.x(\lambda y.y)$
   (b) (2 points) $\lambda x.xx$
   (c) (2 points) $\mathbf{fst}(\mathbf{left}(()))$
   (d) (2 points) $\mathbf{abort}(\lambda x.x)$
   (e) (2 points) $\lambda x.\mathbf{case}(x)\mathbf{of}\{\mathbf{left}(y) \to y(\lambda x.x); \mathbf{right}(z) \to z\}$

2. Type inhabitation problem is the problem of finding a term for a given type under the empty typing context. For example, there is a term that inhabits the type $A \to A$, e.g, we have $\vdash \lambda x.x : A \to A$. Whereas there is no term that inhabits the empty type $\bot$, i.e., we can not find a term M such that $\vdash M : \bot$.

   Consider the following types, determine if there is a term that inhabits the type. If a type is inhabitable, then provide the term and the typing derivation; if a type is not inhabitable, explain why.

   (a) (2 points) $A \to B \to (A \times B)$
   (b) (2 points) $(A \to B) \to (B \to C) \to (A \to C)$
   (c) (2 points) $(A \to C) \to C$
   (d) (2 points) $((A \to \bot) + B) \to (A \to B)$
   (e) (2 points) $(A \to B) \to ((A \to \bot) + B)$