* Functor $F : \underline{C} \to \underline{D}$.

$\underline{C}, \underline{D}$ are categories.

① $\forall$ object $A \in \underline{C}$,

$FA \in obj(\underline{D})$.

② $\forall$ $f : A \to B$ morphism in $\underline{C}$.

$Ff : FA \to FB$. a morphism in $\underline{D}$.

③ $F\, id_A = id_{FA}$

$F(f \circ g) = Ff \circ Fg$.

* $Id_{\underline{C}} : \underline{C} \to \underline{C}$ the identity Functor.

$Id_{\underline{C}}(A) = A$.

$Id_{\underline{C}}\, f = f$

* Functor are closed under composition.

i.e. $F : \underline{C} \to \underline{D}$, $G : \underline{D} \to \underline{E}$.

$G \circ F : \underline{C} \to \underline{E}$.

* Example: Powerset Fuctor $P : \underline{Set} \to \underline{Set}$.

$P(X) = \{ S \mid S \subseteq X \}$.

$\forall f : X \to Y$,

$P(f) : P(X) \to P(Y)$

$P(f)(S) = f(S)$.

$P(id_X)(S) = S$, so $P(id_X) = id_{P(X)}$

$f : X \to Y$, $g : Y \to Z$

$P(g \circ f) : P(X) \to P(Z)$

$P(g \circ f)(S) = g \circ f(S) = g(f(S))$.

on the other hand

$$(P(g)\circ P(f))(S) = P(g)(P(f)(S)) = g(f(S)).$$

* Natural transformation

Let $F, G : \underline{C} \to \underline{D}$ be functors.

a natural transformation $\alpha : F \to G$ consists of

$\forall A \in \underline{C} \quad \alpha_A : FA \to GA \in \underline{D}$

such that

$\forall\ A \xrightarrow{f} B \in \underline{C}$. (naturality)

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

* Functor category.

$$[\underline{C}, \underline{D}] \quad F \in obj([\underline{C}, \underline{D}])$$

$$F : \underline{C} \to \underline{D}.$$

morphisms are natural transformation

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

$$\forall f : A \to B \qquad
\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & GA \arrow[r, "\beta_A"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & HA \arrow[d] \\
FB \arrow[r, "\alpha_B"] & GB \arrow[r, "\beta_B"'] & HB
\end{tikzcd}$$

* $[\underline{C}, \underline{C}]$

A monoid in $[\underline{C}, \underline{C}]$ is called <u>Monad</u>. In other words.

A monad consists of :

$$\mu : M \circ M \to M$$

$$\eta : Id \to M.$$