* Monoid.

Let X be a set, a monoid is a tuple $(X, e, *)$ such that

$e \in X$.

$* : X \times X \to X$ and

called unit.

$\forall a \in X. \quad a * e = e * a = a$.

(assoc) $\forall a, b, c \in X, \quad (a * b) * c = a * (b * c)$

* Examples of Monoid.

$(\mathbb{N}, 1, *)$, $(\mathbb{N}, 0, +)$

$(Y \Rightarrow Y, id_Y, \circ)$,

where $Y \Rightarrow Y = \{ f : Y \to Y \}$

* Monoid homomorphism $(X, e, *) \to (X', e', *')$

$f : X \to Y$ function s.t. $f(e) = e'$

$\forall a b \in X, \quad f(a * b) = f(a) *' f(b)$

* Note that the unit in a monoid is unique.

Claim: Let $(X, *, e)$ be a monoid. if $\exists b \in X$ s.t. $\forall a \in X$ $a * b = b * a = a$.

then $b = e$.

Pf.

$b * e = e$ by assumption

$b * e = b$ by property of e.

so $e = b$. Q.E.D.

* Note: In monoid homomorphism,

$h(a * b) = h(a) *' h(b) \not\Rightarrow h(e) = e'$

Even though $h(a * e) = h(a) *' h(e)$

$= h(a)$.

$\Rightarrow h(a) = h(a) *' h(e) \quad \forall a \in X$.

but it does not imply $\forall b \in X'$,

$b *' h(e) = b$. (h is not sujective)

* Category

A category $\underline{C}$ consist of the following.

① A collection/set of "objects" $obj(\underline{C})$.

② $\forall A, B \in obj(\underline{C})$, a set of "morphisums" $Hom_{\underline{C}}(A, B)$.

for $f \in Hom_{\underline{C}}(A, B)$, we often write $f: A \to B$ for a morphism.

(composition)

③ $\forall f: A \to B$, $g: B \to C$.

$$\exists\ g \circ f: A \to C.$$

moreover

$$\forall h: C \to D.$$

(assoc) $h \circ (g \circ f) = (h \circ g) \circ f$

④ $\forall A \in obj(\underline{C})$.

$\exists\ id_A: A \to A$ s.t.

$\forall f: A \to B$. $f \circ id_A = f$

and $\forall\, g : C \to A$,

$$\mathrm{id}_A \circ g = g.$$

* Examples.

① **Set** objects are sets.

$X \in \underline{\mathrm{Set}}$.

morphisms are functions.

$f : X \to Y$.

compositions are given by function composition.

② **Rel** objects are sets.

morphisms are relations.

$f \in \mathrm{Hom}_{\mathrm{Rel}}(A, B)$ if $f \subseteq A \times B$.

we denote f as $f : A \nrightarrow B$.

$\forall\ f: A \nrightarrow B,$

$g: B \nrightarrow C,$

we define $g \circ f : A \nrightarrow C$ as.

$$= \{ (a,c) \mid a \in A,\ c \in C,\ \exists b \in B \text{ s.t. } (a,b) \in f,\ (b,c) \in g \}$$

$$id_A = \{ (a,a) \mid a \in A \}.$$

③. Let $\underline{C}$ be a category with a single object X.

Then $(Hom_{\underline{C}}(X,X), \circ, id_X)$

is a monoid.

④ $\underline{Mon}$ → category of monoids.

objects are monoids.

morphisms are monoid homomorphism.

* Functor. $F : \underline{C} \to \underline{D}$.

$\underline{C}$, $\underline{D}$ are categories.

① $\forall$ object $A \in \underline{C}$,

$FA \in obj(\underline{D})$.

② $\forall$ $f : A \to B$ morphism in $\underline{C}$.

$Ff : FA \to FB$. a morphism in $\underline{D}$.

③ $F\,id_A = id_{FA}$

$F(f \circ g) = Ff \circ Fg$.