A biset-enriched categorical model for Proto-Quipper with dynamic lifting

Frank Fu

Joint work with K. Kishida, N.J. Ross and P. Selinger

QPL 2022

Motivation

Construct a concrete categorical model for dynamic lifting.

- Circuit generation time and circuit execution time.
- Dynamic lifting.

Categories for the two runtimes

$$\begin{tikzcd} \mathbf{M} \arrow[d, "J"] \\ \mathbf{Q} \end{tikzcd}$$

- $\mathbf{M}$ and $\mathbf{Q}$ are symmetric monoidal categories.
- $\mathrm{obj}(\mathbf{M}) = \mathrm{obj}(\mathbf{Q})$.
- $\mathbf{Q}$ has a coproduct $\mathbf{Bit} = I + I$.
- $\mathbf{Q}$ is enriched in convex spaces.
- The functor J is identity on objects and strict symmetric monoidal.

Modeling dynamic lifting

$$\begin{tikzcd} \mathbf{Bool} \arrow[r, "\mathsf{init}"] \arrow[dr, "\eta"'] & \mathbf{Bit} \arrow[d, "\mathsf{dynlift}"] \\ & T\mathbf{Bool} \end{tikzcd}$$

- T is a commutative monad.

How to combine $\mathbf{M}$ and $\mathbf{Q}$?

$$\begin{tikzcd} \mathbf{M} \arrow[d, "J"] \\ \mathbf{Q} \end{tikzcd}$$

Our answer: biset-enrichment

- The category of *bisets*, i.e., $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$.
- Objects: (X_0, X_1, f).

$$\begin{tikzcd} X_1 \arrow[d, "f"] \\ X_0 \end{tikzcd}$$

- Morphisms: (h_0, h_1).

$$\begin{tikzcd} X_1 \arrow[r, "h_1"] \arrow[d, "f"] & Y_1 \arrow[d, "g"] \\ X_0 \arrow[r, "h_0"] & Y_0 \end{tikzcd}$$

The category of bisets: $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$

- It is cartesian closed, complete and cocomplete.
- $U_0(X_0, X_1, f) = X_0 : \mathbf{Set}^{\mathbf{2}^{\mathrm{op}}} \longrightarrow \mathbf{Set}$.
- $\Delta(X) = (X, X, \mathrm{Id}) : \mathbf{Set} \longrightarrow \mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$.
- $U_0 \dashv \Delta$
- A commutative monad $T = \Delta \circ U_0$.

$\mathcal{V}$-enriched categories

Let $\mathcal{V}$ be a monoidal category. A $\mathcal{V}$-enriched category $\mathbf{A}$ is given by the following:

- A class of objects, also denoted $\mathbf{A}$.
- For any $A, B \in \mathbf{A}$, an object $\mathbf{A}(A, B)$ in $\mathcal{V}$.
- For any $A \in \mathbf{A}$, a morphism $u_A : I \to \mathbf{A}(A, A)$ in $\mathcal{V}$.
- For any $A, B, C \in \mathbf{A}$, a morphism $c_{A,B,C} : \mathbf{A}(A, B) \otimes \mathbf{A}(B, C) \to \mathbf{A}(A, C)$ in $\mathcal{V}$.
- u and c must satisfy suitable diagrams in $\mathcal{V}$.

The biset-enriched category $\mathbf{C}$

- $\mathrm{obj}(\mathbf{C}) = \mathrm{obj}(\mathbf{Q}) = \mathrm{obj}(\mathbf{M})$.
- For any $A, B \in \mathbf{C}$, the hom-object $\mathbf{C}(A, B)$ is a biset:

$$\begin{tikzcd} \mathbf{M}(A,B) \arrow[d, "J_{AB}"] \\ \mathbf{Q}(A,B) \end{tikzcd}$$

The biset-enriched category $\mathbf{C}$

A global map $f : 1 \to \mathbf{C}(A, B)$ is the following.

$$\begin{tikzcd} 1 \arrow[r, "f_1"] \arrow[dr, "f_0"'] & \mathbf{M}(A, B) \arrow[d, "J_{AB}"] \\ & \mathbf{Q}(A, B) \end{tikzcd}$$

The biset-enriched category $\overline{\mathbf{C}}$

Define $\overline{\mathbf{C}} := \mathcal{V}^{\mathbf{C}^{\mathrm{op}}}$, where $\mathcal{V} = \mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$.

- $\overline{\mathbf{C}}$ is monoidal closed.
- $\mathbf{Bit} = y\mathbf{Bit} = \mathbf{C}(-, \mathbf{Bit})$.
- $\mathbf{Bool} = yI + yI = \mathbf{C}(-, I) + \mathbf{C}(-, I)$.
- A commuative $\mathcal{V}$-monad $\overline{T}(F) = T \circ F$.

The biset-enriched category $\overline{\mathbf{C}}$

$\overline{\mathbf{C}} := \mathcal{V}^{\mathbf{C}^{\mathrm{op}}}$, where $\mathcal{V} = \mathbf{Set}^{\mathbf{2}^{\mathbf{op}}}$.

$$\begin{tikzcd}
\mathbf{Bool} \arrow[r, "\mathsf{init}"] \arrow[dr, "\eta"'] & \mathbf{Bit} \arrow[d, "?"] \\
& \overline{T}\mathbf{Bool}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Q}(-, I) + \mathbf{Q}(-, I) \arrow[r] \arrow[dr, "\mathrm{Id}"'] & \mathbf{Q}(-, I + I) \arrow[d, "?"] \\
& \mathbf{Q}(-, I) + \mathbf{Q}(-, I)
\end{tikzcd}$$

The biset-enriched category $\overline{\mathbf{C}}$

$\overline{\mathbf{C}} := \mathcal{V}^{\mathbf{C}^{\mathrm{op}}}$, where $\mathcal{V} = \mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$.

$$
\begin{tikzcd}
\mathbf{Bool} \arrow[r, "\mathsf{init}"] \arrow[dr, swap, "\eta"] & \mathbf{Bit} \arrow[d, "?"] \\
& \overline{T}\mathbf{Bool}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{Q}(-, I) + \mathbf{Q}(-, I) \arrow[r] \arrow[dr, swap, "\mathrm{Id}"] & \mathbf{Q}(-, I + I) \arrow[d, "?"] \\
& \mathbf{Q}(-, I) + \mathbf{Q}(-, I)
\end{tikzcd}
$$

Yoneda embedding does not preserve coproducts!

The subcategory $\widetilde{\mathbf{C}}$

- For every $\mathcal{V}$-functor $F : \mathbf{C} \to \mathcal{V} \in \widetilde{\mathbf{C}}$, there is an ordinary functor $F^0 : \mathbf{Q} \to \mathbf{Set}$
- Define $\widetilde{\mathbf{C}}$ to be the full subcategory of $\overline{\mathbf{C}}$, where for every $F \in \widetilde{\mathbf{C}}$, the functor $F^0 : \mathbf{Q} \to \mathbf{Set}$ is *product-preserving*.
- We call the embedding $\overline{y} : \mathbf{C} \to \widetilde{\mathbf{C}}$ *Lambek embedding*.

$$\begin{tikzcd} \mathbf{C} \arrow[d, "\overline{y}"] \arrow[dr, "y"] & \\ \widetilde{\mathbf{C}} \arrow[r, hook] & \overline{\mathbf{C}} \end{tikzcd}$$

The subcategory $\widetilde{\mathbf{C}}$

$\widetilde{\mathbf{C}}$ satisfies all the requirements for being a model for dynamic lifting.

- $\widetilde{\mathbf{C}}$ is a reflective subcategory of $\overline{\mathbf{C}}$.
- $\widetilde{\mathbf{C}}$ is symmetric monoidal closed.
- $\widetilde{\mathbf{C}}$ has a commutative monad $\widetilde{T}$.
- $\widetilde{\mathbf{C}}$ has dynamic lifting morphism $\mathsf{dynlift} : \mathbf{Bit} \to \widetilde{T}\mathbf{Bool}$.

Thank you!