A Framework for Internalizing Relations into Type Theory

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Introduction: Motivations

- Common pattern on some typing rules:

$$\frac{\Gamma \vdash t : T \quad T <: T'}{\Gamma \vdash t : T'}\ \mathit{sub} \qquad \frac{\Gamma \vdash t : T \quad T \cong T'}{\Gamma \vdash t : T'}\ \mathit{conv}$$

- Typing appeals to external judgments (relations).
- Can we find a general way to internalize these relations as types?
- Benefit: Enable hypothetical reasoning. e.g. $x : T < T' \vdash t : T''$.

Introduction: Outline of the Development

- Start from a *base system*: dependent variant of Curry-Style System F.
- Build up an *internalized system*: system with internalized relations.
- Girard-Tait reducibility method to show normalization.

Introduction: Major Results

- We define *internalization structure* to encapsulate the additional information we need to build an internalized system.
- We show internalized system preserves the normalization of the base system in the closed term environment.
- We present three examples to show the utility of this methodology.

Base System: Syntax

$$\begin{array}{lll} \textbf{Types } T & ::= & X \mid B \mid \Pi x : T_1 . T_2 \mid \forall X . T \\ \textbf{Terms } t & ::= & \textbf{axiom} \mid x \mid (t_1 \ t_2) \mid \lambda x . t \\ \textbf{Values } v & ::= & \lambda x . t \mid \textbf{axiom} \\ \textbf{Context } \Gamma & ::= & \cdot \mid \Gamma, x : T \mid \Gamma, X \end{array}$$

Operational Semantics: CBN.

Base System: Type Assignment

$$\frac{(x:T)\in\Gamma \quad \Gamma\vdash \textbf{OK}}{\Gamma\vdash x:T}\ \mathit{Var}$$

$$\frac{\Gamma, x:T_1 \ \vdash t:T_2}{\Gamma\vdash \lambda x.t:\Pi x:T_1.T_2}\ \Pi\text{-}\mathit{intro}$$

$$\frac{\Gamma\vdash t_1:\Pi x:T_1.T_2 \quad \Gamma\vdash t_2:T_1}{\Gamma\vdash t_1\ t_2:[t_2/x]T_2}\ \Pi\text{-}\mathit{elim}$$

$$\frac{\Gamma, X\vdash t:T}{\Gamma\vdash t:\forall X.T}\ \forall\text{-}\mathit{intro}$$

$$\frac{\Gamma\vdash t:\forall X.T \quad \mathit{FVar}(T')\subseteq \mathit{dom}(\Gamma)}{\Gamma\vdash t:[T'/X]T}\ \forall\text{-}\mathit{elim}$$

Base System: Reducibility Candidate

Definition

A reducibility candidate $\mathcal{R}$ is a set of terms that satisfies the following conditions:

CR 1 If $t \in \mathcal{R}$,then $t \in \mathcal{V}$, where $\mathcal{V}$ is the set of closed terms that reduces to a value .
CR 2 If $t \in \mathcal{R}$ *and* $t \rightsquigarrow t'$, then $t' \in \mathcal{R}$.
CR 3 If t is a closed term, $t \rightsquigarrow t'$ *and* $t' \in \mathcal{R}$, then $t \in \mathcal{R}$.

Base System: Reducibility Sets

Let $\Re$ be the set of all reducibility candidates. ϕ is a mapping between free type variables and reducibility candidates.

$$
\begin{array}{lll}
t \in [\![B]\!]_\phi & \text{iff} & t \in \mathcal{R}_B, \text{ where } \mathcal{R}_B \in \Re \\
t \in [\![X]\!]_\phi & \text{iff} & t \in \phi(X) \\
t \in [\![\Pi x : T_1 . T_2]\!]_\phi & \text{iff} & t \in \mathcal{V} \text{ and } (\forall u \in [\![T_1]\!]_\phi \Rightarrow (t\ u) \in [\![[u/x]T_2]\!]_\phi) \\
t \in [\![\forall X . T]\!]_\phi & \text{iff} & \forall \mathcal{R} \in \Re, t \in [\![T]\!]_{\phi[\mathcal{R}/X]}
\end{array}
$$

Theorem (Type Soundness)

If $\Gamma \vdash t : T$, *then* $\forall(\sigma, \phi) \in [\Gamma], (\sigma\ t) \in [\![\sigma T]\!]_\phi$.

Corollary

If $\cdot \vdash t : T$, *then t is normalizing.*

Internalized System: Syntax

RTypes	A	$::=$	$R^{(n\times m)}(t_1, ..., t_n, T_1, ..., T_m)$
ETypes	S	$::=$	$B \mid X \mid \Pi x : S.S \mid \forall X.S \mid A$
EContext	Δ	$::=$	$\cdot \mid \Delta, x : S \mid \Delta, X$
Terms	t	$::=$	$\textbf{axiom} \mid x \mid \lambda x.t \mid t_1\, t_2$

Note:

- $R^{n\times m}$ is a symbol represents external relations.
- Term **axiom** will introduce $R^{(n\times m)}(t_1, ..., t_n, T_1, ..., T_m)$.

Internalized System: Typing Rules(I)

$$\frac{A \in D \quad FVar(A) \subseteq dom(\Delta) \quad \Delta \vdash \mathbf{OK}}{\Delta \vdash \mathbf{axiom} : A} \; A\text{-}intro$$

$$\frac{\Delta \vdash t : T \quad \Delta \vdash t' : A \quad E(\Delta, t, t', A, T, T')}{\Delta \vdash t : T'} \; A\text{-}elim$$

$$\frac{\Delta \vdash t : \forall X.S \quad [S'/X]S \in \mathbf{ETypes} \quad FVar(S') \subseteq dom(\Delta)}{\Delta \vdash t : [S'/X]S} \; \forall\_elim$$

Notice: $(T < T') < T \notin \mathbf{ETypes}$!

Internalized System: Typing Rules(II)

$$\frac{\Delta, X \vdash t : S}{\Delta \vdash t : \forall X.S}\ \forall\_intro$$

$$\frac{\Delta(x) = S \quad \Delta \vdash \mathbf{OK}}{\Delta \vdash x : S}\ Var$$

$$\frac{\Delta, x : S_1 \ \vdash t : S_2}{\Delta \vdash \lambda x.t : \Pi x : S_1.S_2}\ \Pi\_intro$$

$$\frac{\Delta \vdash t_1 : \Pi x : S_1.S_2 \quad \Delta \vdash t_2 : S_1}{\Delta \vdash t_1\ t_2 : [t_2/x]S_2}\ \Pi\_elim$$

Internalization Structure:$\langle D, E, \mathcal{I} \rangle$

- *Reflective Relational Sentences*: $D \subseteq$ **RTypes**.
- *Elimination Relation*:
  $E \subseteq$ **EContext** $\times$ **Terms** $\times$ **Terms** $\times$ **RTypes** $\times$ **ETypes** $\times$ **ETypes**.
- *Interpretation* of Relation $R^{(n \times m)}$: $\mathcal{I}_{R^{(n\times m)}} \subseteq \textbf{Terms}^n \times \Re^m$.

$$
\begin{array}{llll}
\textbf{RTypes} & A & ::= & R^{(n\times m)}(t_1, ..., t_n, T_1, ..., T_m) \\
\textbf{ETypes} & S & ::= & B \mid X \mid \Pi x : S.S \mid \forall X.S \mid A \\
\textbf{Terms} & t & ::= & \textbf{axiom} \mid x \mid \lambda x.t \mid t_1\, t_2
\end{array}
$$

Internalization Structure: Interpretation of the Extended Types

Definition

$\mathcal{A}$ is the set of closed terms that normalize at **axiom**.

- $t \in [\![B]\!]_\phi$ iff $t \in \mathcal{R}_B$.
- $t \in [\![X]\!]_\phi$ iff $t \in \phi(X)$.
- $t \in [\![\Pi x : S_1.S_2]\!]_\phi$ iff $t \in \mathcal{V}$ and $(\forall u \in [\![S_1]\!]_\phi \Rightarrow (t\ u) \in [\![[u/x]S_2]\!]_\phi)$.
- $t \in [\![\forall X.S]\!]_\phi$ iff $\forall \mathcal{R} \in \Re, t \in [\![S]\!]_{\phi[\mathcal{R}/X]}$.
- $t \in [\![R^{(n\times m)}(t_1, ..., t_n, T_1, ..., T_m)]\!]_\phi$ iff $t \in \mathcal{A}$ and $(t_1, ..., t_n, [\![T_1]\!]_\phi, ..., [\![T_m]\!]_\phi) \in \mathcal{I}_{R^{(n\times m)}}$.

Internalization Structure: Soundness Properties

- **Soundness of reflective relational sentences**

$$\frac{R^{(n\times m)}(t_1, ..., t_n, T_1, ..., T_m) \in D \quad (\sigma, \phi) \in [\Delta]}{\forall \phi, \forall \sigma \in Sub, (\sigma t_1, ..., \sigma t_n, [\![\sigma T_1]\!]_\phi, ..., [\![\sigma T_m]\!]_\phi) \in \mathcal{I}_{R^{(n\times m)}}}$$

- **Soundness of the elimination relation**

$$\frac{\sigma(t) \in [\![\sigma S]\!]_\phi \quad (\Delta, t, t', R^{(n\times m)}(t_1, ..., t_n, T_1, ..., T_m), S, S') \in E}{\sigma(t) \in [\![\sigma S']\!]_\phi}$$

Internalization Structure: Normalization

Theorem (Type Soundness)

If $\langle D, E, \mathcal{I} \rangle$ is an internalization structure and $\Delta \vdash t : S$, then $\forall(\sigma, \phi) \in [\Delta], (\sigma\ t) \in [\![\sigma S]\!]_{\phi}$.

Corollary

If $\langle D, E, \mathcal{I} \rangle$ is an internalization structure and $\cdot \vdash t : S$, then t is normalizing.

Note:

- $y : (\top \to \top) < \top \vdash (\lambda x.xx)\lambda x.xx : \top$.
- CBN and CBV.

Example: Subtyping

- Syntax:

RTypes	A	$::=$	$T < T'$
ETypes	S	$::=$	$\top \mid \bot \mid X \mid \Pi x : S.S \mid \forall X.S \mid A$
Terms	t	$::=$	**axiom** $\mid x \mid \lambda x.t \mid t_1 t_2$

- Reflective relational sentences D:

$$\frac{}{T < \top \in D} \qquad \frac{}{\bot < T \in D}$$

$$\frac{}{X < X \in D} \qquad \frac{T_1 < T_2 \in D}{\forall X.T_1 < \forall X.T_2 \in D}$$

$$\frac{T_1' < T_1 \in D \quad T_2 < T_2' \in D}{\Pi x : T_1.T_2 < \Pi x : T_1'.T_2' \in D}$$

Example: $\langle D, E, \mathcal{I}_{<^{0\times 2}} \rangle$

- Elimination Relation E:

  $$(\Delta, t, t', T < T', T, T') \in E.$$

- Interpretation $\mathcal{I}_{<}$:

  $$\mathcal{I}_{<} := \{(\mathcal{R}_1, \mathcal{R}_2) \mid \mathcal{R}_1 \subseteq \mathcal{R}_2\}$$

- Soundness Properties:
  - If $(T < T') \in D$, then $\forall \sigma \in \mathit{Sub}, \forall \phi, ([\![\sigma T]\!]_{\phi}, [\![\sigma T']\!]_{\phi}) \in \mathcal{I}_{<}$.
  - If $(\Delta, t, t', T_1 < T_2, T_1, T_2) \in E$, $(\sigma, \phi) \in [\Delta]$ and $\sigma(t) \in [\![\sigma T_1]\!]_{\phi} = [\![T_1]\!]_{\phi}$ and $T_1 < T_2 \in D$, then $\sigma(t) \in [\![\sigma T_2]\!]_{\phi} = [\![T_2]\!]_{\phi}$.

Example: Internalized System for Subtyping

For this internalization structure–$\langle D, E, \mathcal{I}_{<} \rangle$, the *A-intro* and *A-elim* rules are equivalent to:

$$\frac{T_1 < T_2 \in D \quad FVar(T_1 < T_2) \subseteq dom(\Delta) \quad \Delta \vdash \mathbf{OK}}{\Delta \vdash \mathbf{axiom} : T_1 < T_2} \text{ A-intro}$$

$$\frac{\Delta \vdash t : T_1 \quad \Delta \vdash t' : T_1 < T_2}{\Delta \vdash t : T_2} \text{ A-elim}$$

Notice: This appears more expressive than $F_{<:}$, since $\forall X <: T.T'$ is $\forall X.\Pi u : X <: T.T'$.

Conclusions

- Futrue works:
  - For two other examples(term joinability and closed-term type inhabitation), please see the paper.
  - Prove type preservation of the internalized system?
  - Find out more relations that can be incorporated into base system through internalization.
  - Explore the flexibility of this method more.
- I want to say thank you for listening to this talk.
- Last but not least, I thank the PSATTT reviewers.