A biset-enriched categorical model for Proto-Quipper with dynamic lifting

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Motivation

Categorical semantics for Quipper and Proto-Quipper.

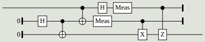
- Circuit generation time and circuit execution time.
- Dynamic lifting.

Quipper/Proto-Quipper's two runtimes

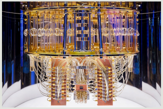
```
alice2 : !(Qubit -> Qubit -> Bool * Bool)
alice2 a q =
  let (a, q) = CNot a q
      q = H q
  in (dynlift (Meas a), dynlift (Meas q))

bob2 : !(Qubit -> Bool -> Bool -> Qubit)
bob2 q x y =
  let q = if x then QNot q else q
      q = if y then ZGate q else q
  in q
```

Circuit generation time



Circuit execution time



Dynamic lifting

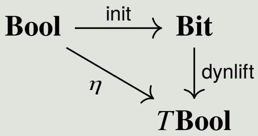
- *Parameters*: circuit generation time values (e.g., **Bool**, **Nat**)
- *States*: circuit execution time values (e.g., **Bit**, **Qubit**).
- Measurement: **Qubit** $\rightarrow$ **Bit**.
- Dynamic lifting: an operation that lifts a **Bit** to a **Bool**.

Categories for the two runtimes

$$\begin{array}{c} \mathbf{M} \\ \big\downarrow{\scriptstyle J} \\ \mathbf{Q} \end{array}$$

- **M** represents the category of quantum circuits.
- **Q** represents the category of quantum operations.
- The *interpretation* functor J.

Modeling dynamic lifting



T is a strong commutative monad.

How to combine **M** and **Q**?

$$\begin{array}{c}\mathbf{M} \\ \big\downarrow J \\ \mathbf{Q}\end{array}$$

Our answer: *biset-enrichment*

- The category of *bisets*, i.e., $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$.
- Objects: (X_0, X_1, f).

$$\begin{array}{c} X_1 \\ \big\downarrow f \\ X_0 \end{array}$$

- Morphisms: (h_0, h_1).

$$\begin{array}{ccc} X_1 & \xrightarrow{h_1} & Y_1 \\ \big\downarrow f & & \big\downarrow g \\ X_0 & \xrightarrow{h_0} & Y_0 \end{array}$$

The category of bisets: $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$

- It is cartesian closed, complete and cocomplete.
- $U_0(X_0, X_1, f) = X_0 : \mathbf{Set}^{\mathbf{2}^{\mathrm{op}}} \to \mathbf{Set}$.
- $\Delta(X) = (X, X, \mathrm{Id}) : \mathbf{Set} \to \mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$.
- $U_0 \dashv \Delta$
- A strong commutative monad $T = \Delta \circ U_0$.

The biset-enriched category $\mathbf{C}$

- $\mathrm{obj}(\mathbf{C}) = \mathrm{obj}(\mathbf{Q}) = \mathrm{obj}(\mathbf{M})$.
- For any $A, B \in \mathbf{C}$, the hom-object $\mathbf{C}(A, B)$ is a biset:

$$\begin{array}{c} \mathbf{M}(A, B) \\ \big\downarrow J_{AB} \\ \mathbf{Q}(A, B) \end{array}$$

The biset-enriched category **C**

A global map $f : 1 \to \mathbf{C}(A, B)$ is the following.

$$\begin{array}{ccc} 1 & \xrightarrow{f_1} & \mathbf{M}(A, B) \\ & {}_{f_0}\searrow & \downarrow J_{AB} \\ & & \mathbf{Q}(A, B) \end{array}$$

The biset-enriched category $\overline{\mathbf{C}}$

Define $\overline{\mathbf{C}} := \mathcal{V}^{\mathbf{C}^{\mathrm{op}}}$, where $\mathcal{V} = \mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$.

- $\overline{\mathbf{C}}$ is monoidal closed.
- $\mathbf{Bit} = y\mathbf{Bit} = \mathbf{C}(-, \mathbf{Bit})$.
- $\mathbf{Bool} = yI + yI = \mathbf{C}(-, I) + \mathbf{C}(-, I)$.
- A strong commuative $\mathcal{V}$-monad $\overline{T}(F) = T \circ F$.

The biset-enriched category $\overline{\mathbf{C}}$

$\overline{\mathbf{C}} := \mathcal{V}^{\mathbf{C}^{\mathrm{op}}}$, where $\mathcal{V} = \mathbf{Set}^{\mathbf{2}^{\mathbf{op}}}$.

$$\begin{array}{ccc}
\mathbf{Bool} & \xrightarrow{\ \mathsf{init}\ } & \mathbf{Bit} \\
& \searrow^{\eta} & \downarrow ? \\
& & \overline{T}\mathbf{Bool}
\end{array}$$

$$\begin{array}{ccc}
\mathbf{Q}(-, I) + \mathbf{Q}(-, I) & \longrightarrow & \mathbf{Q}(-, I + I) \\
& \searrow^{\mathrm{Id}} & \downarrow ? \\
& & \mathbf{Q}(-, I) + \mathbf{Q}(-, I)
\end{array}$$

The biset-enriched category $\overline{\mathbf{C}}$

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Yoneda embedding does not preserve coproducts!

The subcategory $\widetilde{\mathbf{C}}$

- For every $\mathcal{V}$-functor $F : \mathbf{C} \to \mathcal{V} \in \overline{\mathbf{C}}$, there is an ordinary functor $F^0 : \mathbf{Q} \to \mathbf{Set}$
- Define $\widetilde{\mathbf{C}}$ to be the full subcategory of $\overline{\mathbf{C}}$, where for every $F \in \widetilde{\mathbf{C}}$, the functor $F^0 : \mathbf{Q} \to \mathbf{Set}$ is *product-preserving*.

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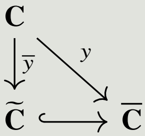
- Define $\widetilde{\mathbf{C}}$ to be the full subcategory of $\overline{\mathbf{C}}$, where for every $F \in \widetilde{\mathbf{C}}$, the functor $F^0 : \mathbf{Q} \to \mathbf{Set}$ is *product-preserving*.
- We call the embedding $\overline{y} : \mathbf{C} \to \widetilde{\mathbf{C}}$ *Lambek embedding*, which preserves coproducts.



The subcategory $\widetilde{\mathbf{C}}$ is a model for dynamic lifting

- $\widetilde{\mathbf{C}}$ is a reflective subcategory of $\overline{\mathbf{C}}$.
- $\widetilde{\mathbf{C}}$ is symmetric monoidal closed.
- $\widetilde{\mathbf{C}}$ has a strong commutative monad $\widetilde{T}$.
- $\widetilde{\mathbf{C}}$ has dynamic lifting morphism $\mathsf{dynlift} : \mathbf{Bit} \to \widetilde{T}\mathbf{Bool}$.

Thank you!