Strong PV Numbers of Degree 4

Relating to λ-convex and λ-clonvex Sets

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λ Operations and Sets

Definition: Fix a number $\lambda \in \mathbb{C}$. For any $a, b \in \mathbb{C}$, define the operation

$$a \star b := (1-\lambda)a + \lambda b$$

For any set $S \subseteq \mathbb{C}$

1. We say that S is *λ-convex* iff for every $a, b \in S$, the point $a \star b$ is in S.
2. We say that S is *λ-convex closed* (or *λ-clonvex*) iff S is λ-convex and topologically closed.

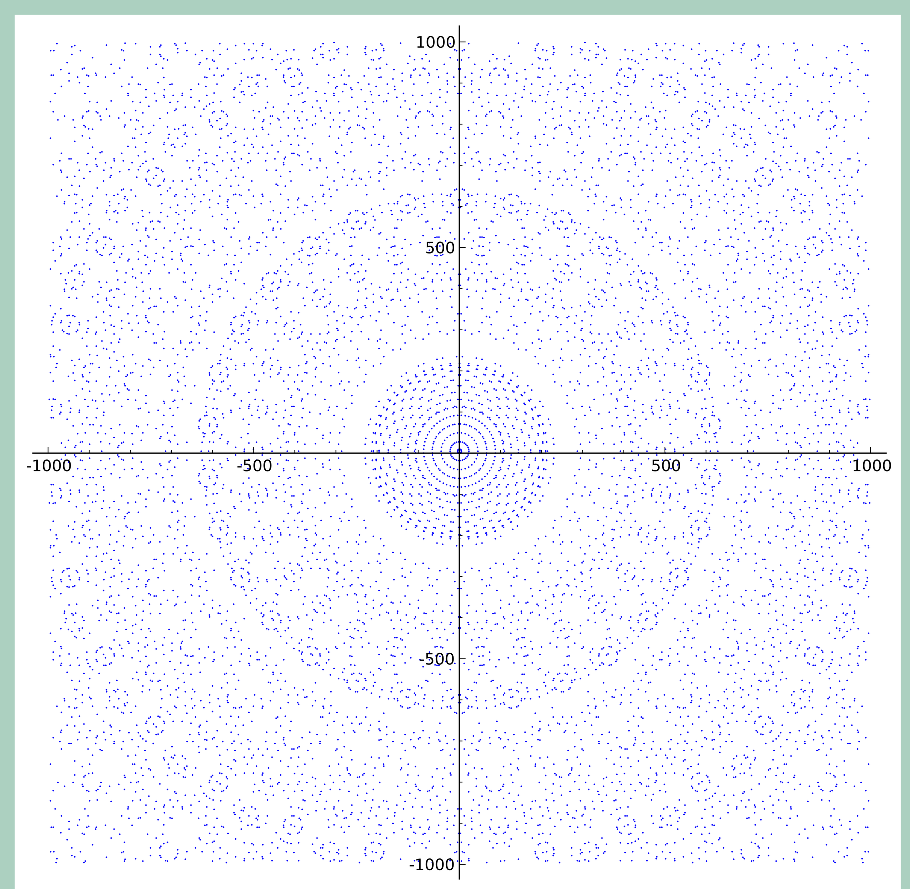
Definition: For any $\lambda \in \mathbb{C}$ and any set $S \subseteq \mathbb{C}$

1. $Q_\lambda(S)$ is defined to be the $\subseteq$-minimum λ-convex superset of S, or the *λ-convex closure* of S
2. $R_\lambda(S)$ is defined to be the $\subseteq$-minimum λ-clonvex superset of S, or the *λ-clonvex closure*.

In both cases, if the set S is omitted then S is taken to to be the set $\{0, 1\}$

R_λ, Q_λ, and Quasicrystals

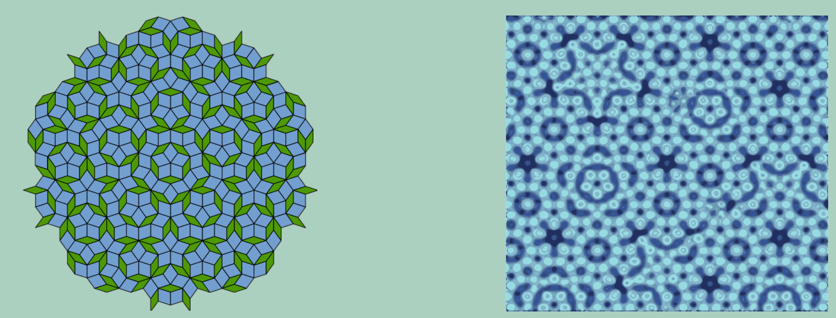
When λ is Strong PV, then R_λ is discrete and closed, so $R_\lambda = Q_\lambda$. In addition, these form aperiodic "almost lattices" known as Meyer sets.



Visualization of $Q_{\lambda_{15}}(P_{30})$, λ_{15} has minimal polynomial $x^4 - 24x^3 + 26x^2 - 9x + 1$

Abstract

λ-convex and λ-clonvex share lots of connections to periodic and aperiodic Meyer sets. However many pieces of this relationship remain unexplored, including an exact characterization of when these properties arise. Here we seek to characterize strong PV numbers of degree four to further investigate the relationship between λ-convexity and such aperiodic sets.



Penrose tiling and quasicrystals, two applications of Meyer sets

Quartic Strong PV Numbers

Many Q_λ Meyer sets can be generated from strong PV numbers.

Unfortunately, strong PV numbers are not well understood. So far, most research has focused on quadratic polynomials due to their simplicity.

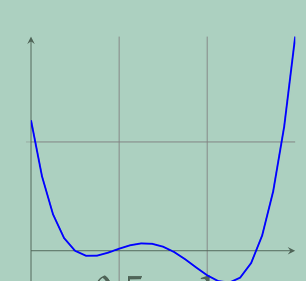
We seek to characterize all strong PV numbers that are roots of quartic polynomials, using the following characteristics:

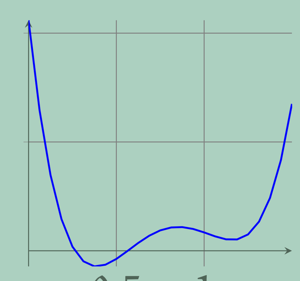
1. Discriminant for a polynomial $p(x) = (x - a_1) \ldots (x - a_d)$ where $a_1 \ldots a_d$ are roots:

$$\Delta := \prod_{i<j} (a_i - a_j)^2$$

The discriminant is positive(negative) if the number of pairs of complex roots is even(odd).

2. Strong PV numbers can be real or complex, which divides the search space into two categories (excluding reflections over the y axis)

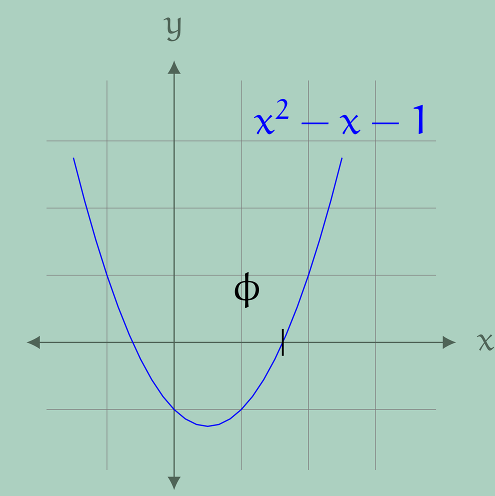




PV and Strong PV Numbers

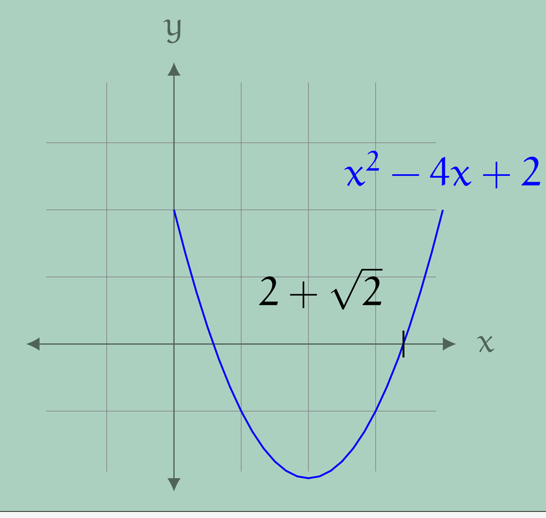
Pisot-Vijayaraghavan (PV) numbers are algebraic numbers where all of the Galois conjugates are smaller than 1 in absolute value. A number is a Strong PV-number iff it is algebraic and all of its Galois conjugates (aside from its complex conjugate) lie in the unit interval.

For example, ϕ is PV since it is algebraic and its conjugate $\phi^{-1} \approx -0.618$ lies in the interval $[-1, 1]$:



It is not strong PV, since its conjugate is negative.

On the other hand, $2+\sqrt{2}$ is Strong PV, since its conjugate $0 < 2-\sqrt{2} < 1$, and $\left[x-(2+\sqrt{2})\right] \cdot \left[x-(2-\sqrt{2})\right] = x^2 - 4x + 2 \in \mathbb{Z}[x]$



Forthcoming Research

- Is there a small (polynomial-sized) classification scheme for all strong PV numbers?
- Can any Meyer set be generated as a λ-clonvex set?

References

[1] Stephen Fenner, Frederic Green, and Rohit Gurjar. Some properties of sets in the plane closed under affine combination by a fixed parameter.