PHYS 704
Homework 5

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March 2, 2016

1. (a) Construct the free-space Green function $G(x, y; x', y')$ for two-dimensional electrostatics by integrating $1/R$ with respect to $(z' - z)$ between the limits $\pm Z$ where Z is taken to be very large. Show that apart from an inessential constant, the Green function can be written alternately as

$$\begin{aligned} G(x, y; x', y') &= -\ln\left[(x - x')^2 + (y - y')^2\right] \\ &= -\ln\left[\varrho^2 + \varrho'^2 - 2\varrho\varrho' \cos(\phi - \phi')\right] \end{aligned}$$

Solution. Use the following definitions:

$$\begin{aligned} \alpha &:= \sqrt{(x - x')^2 + (y - y')^2} \\ u &:= z - z' \end{aligned}$$

to integrate

$$\begin{aligned} \int_{-Z}^{Z} \frac{du}{\sqrt{\alpha^2 + u^2}} &= \ln\left(\sqrt{\alpha^2 + u^2} + u\right)\Big|_{-Z}^{Z} \\ &= \ln \frac{\sqrt{Z^2 + \alpha^2} + Z}{\sqrt{Z^2 + \alpha^2} - Z} \\ &= \ln \frac{\sqrt{1 + \frac{\alpha^2}{Z^2}} + 1}{\sqrt{1 + \frac{\alpha^2}{Z^2}} - 1} \end{aligned}$$

Expanding to first order in $\frac{\alpha^2}{Z^2}$ yields

$$\begin{aligned}
G &= \ln \frac{2 + \frac{\alpha^2}{2Z^2}}{\frac{\alpha^2}{2Z^2}} \\
&= \ln \frac{4Z^2 + \alpha^2}{\alpha^2} \\
&= \ln\left(4Z^2 + \alpha^2\right) - \ln \alpha^2 \\
G(Z \gg \alpha) &= \ln 4Z^2 - \ln \alpha^2
\end{aligned}$$

$$\begin{aligned}
\therefore G(x, y; x', y') &= -\ln \alpha^2 \\
&= -\ln\left[(x - x')^2 + (y - y')^2\right]
\end{aligned}$$

Changing to cylindrical coordinates yields

$$G(x, y; x', y') = -\ln\left[\varrho^2 + \varrho'^2 - 2\varrho\varrho' \cos(\phi - \phi')\right]$$

(b) Show explicitly by separation of variables in polar coordinates that the Green function can be expressed as a Fourier series in the azimuthal coordinate,

$$G = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{im(\phi - \phi')} g_m(\varrho, \varrho')$$

where the radial Green functions satisfy

$$\frac{1}{\varrho'} \frac{\partial}{\partial \varrho'} \left(\varrho' \frac{\partial g_m}{\partial \varrho'}\right) - \frac{m^2}{\varrho'^2} g_m = -4\pi \frac{\delta(\varrho - \varrho')}{\varrho}$$

Note that $g_m(\varrho, \varrho')$ for fixed ϱ is a different linear combination of the solutions of the homogenous radial equation (2.68) for $\varrho' < \varrho$ and for $\varrho' > \varrho$, with a discontinuity of slope at $\varrho' = \varrho$ determined by the source delta function

Solution. The defining equations of the greens function

$$\begin{gathered}
\int_\Omega \nabla'^2 G(\varrho, \phi; \varrho', \phi') \varrho' d\varrho' d\phi' = -4\pi \\
\nabla'^2 G(\varrho, \phi; \varrho', \phi') \propto \delta(\varrho - \varrho')\delta(\phi - \phi')
\end{gathered}$$

can be satisfied if

$$\nabla'^2 G = -4\pi \frac{\delta(\varrho - \varrho')\delta(\phi - \phi')}{\varrho}$$

Applying the laplacian to the given expansion yields:

$$\begin{aligned}
\nabla'^2 G &= \left[\frac{1}{\varrho'}\frac{\partial}{\partial \varrho'}\left(\varrho'\frac{\partial}{\partial \varrho'}\right) - \frac{1}{\varrho'^2}\frac{\partial}{\partial \phi'^2}\right]\left[\frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\varrho, \varrho')\right] \\
&= \frac{1}{2\pi}\sum_{-\infty}^{\infty}\left[\frac{1}{\varrho'}\frac{\partial}{\partial \varrho'}\left(\varrho'\frac{\partial g_m}{\partial \varrho'}\right) - \frac{m^2 g_m}{\varrho'^2}\right] e^{im(\phi-\phi')} \\
&= \frac{1}{2\pi}\sum_{-\infty}^{\infty}\left[-4\pi\frac{\delta(\varrho-\varrho')}{\varrho}\right] e^{im(\phi-\phi')} \\
&= -2\frac{\delta(\varrho-\varrho')}{\varrho}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')} \\
&= -4\pi\frac{\delta(\varrho-\varrho')\delta(\phi-\phi')}{\varrho}
\end{aligned}$$

Showing that the expansion is correct.

(c) Complete the solution and show that the free-space Green function has the expansion

$$G(\varrho, \phi; \varrho', \phi') = -\ln(\varrho_>^2) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\varrho_<}{\varrho_>}\right)^m \cdot \cos\left[m(\phi - \phi')\right]$$

where $\varrho_<$ ($\varrho_>$) is the smaller (larger) of ϱ and ϱ'

Solution. Since $g_m(\varrho, \varrho') = P(\varrho, \varrho')$ is a solution to the (split) Laplace equation, g_m can be given by

$$g_m(\varrho, \varrho') = \begin{cases} A_m \varrho'^m & \varrho' < \varrho \\ B_m \varrho'^{-m} & \varrho' > \varrho \end{cases}$$

Continuity dictates that

$$A_m \varrho^m = B_m \varrho^{-m}$$
$$\therefore A_m = \alpha_m \varrho^{-m}, \quad B_m = \alpha_m \varrho^m$$

Giving

$$g_m(\varrho, \varrho') = \begin{cases} \alpha_m \left(\frac{\varrho'}{\varrho}\right)^m & \varrho' < \varrho \\ \alpha_m \left(\frac{\varrho}{\varrho'}\right)^m & \varrho' > \varrho \end{cases}$$

The discontinuity in the derivative is determined by the laplace equation for the Green function:

$$\begin{aligned} -\frac{4\pi}{\varrho} &= \left.\frac{dg_m}{d\varrho'}\right|_{\varrho<} - \left.\frac{dg_m}{d\varrho'}\right|_{\varrho>} \\ &= -\frac{2m\alpha_m}{\varrho} \\ \therefore \alpha_m &= \frac{2\pi}{m} \end{aligned}$$

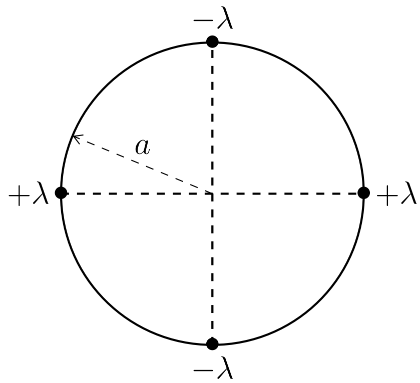
$$\begin{aligned} g_m(\varrho, \varrho') &= \begin{cases} \frac{2\pi}{m} \left(\frac{\varrho'}{\varrho}\right)^m & \varrho' < \varrho \\ \frac{2\pi}{m} \left(\frac{\varrho}{\varrho'}\right)^m & \varrho' > \varrho \end{cases} \\ &= \frac{2\pi}{m}\left(\frac{\varrho_<}{\varrho_>}\right)^m \end{aligned}$$

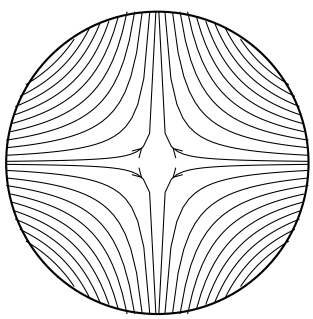
From part a, $g_0 = -\ln \varrho_>^2$, so

$$\begin{aligned} G(\varrho, \varrho') &= -\ln({\varrho_>}^2) + \sum_{m=-\infty, m\neq 0}^{\infty} \frac{1}{|m|}\left(\frac{\varrho_<}{\varrho_>}\right)^{|m|} e^{im(\phi-\phi')} \\ &= -\ln({\varrho_>}^2) + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\varrho_<}{\varrho_>}\right)^m e^{-im(\phi-\phi')} + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\varrho_<}{\varrho_>}\right)^m e^{im(\phi-\phi')} \\ &= -\ln({\varrho_>}^2) + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\varrho_<}{\varrho_>}\right)^m \left(e^{-im(\phi-\phi')} + e^{im(\phi-\phi')}\right) \\ &= -\ln({\varrho_>}^2) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\varrho_<}{\varrho_>}\right)^m \cos\left[m(\phi-\phi')\right] \end{aligned}$$

Where the absolute values preserve the relations on $\varrho_>, \varrho_<$

2. Two dimensional electric quadrupole focusing fields for particle accelerators can be modeled by a set of four symmetrically placed line charges, with linear charge densities $\pm\lambda$ as shown in the left hand figure (the right-hand figure shows the electric field lines)





The charge density in two dimensions can be expressed as

$$\sigma(\varrho, \phi) = \frac{\lambda}{a} \sum_{n=0}^{3} (-1)^n \delta(\varrho - a)\delta\left(\phi - \frac{n\pi}{2}\right)$$

(a) Using the Green function expansion from Problem 2.17c, show that the electrostatic potential is

$$\Phi(\varrho, \phi) = \frac{\lambda}{\pi\epsilon_0} \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{\varrho_<}{\varrho_>}\right)^{4k+2} \cos\left[(4k+2)\phi\right]$$

Solution.

$$\begin{aligned}\Phi(\varrho,\phi) &= \frac{1}{4\pi\epsilon_0}\int_\Omega \sigma(\varrho',\phi')G(\varrho,\phi;\varrho',\phi')\varrho d\varrho d\phi \\ &= \frac{\lambda}{4\pi\epsilon_0}\sum_{n=0}^{3}(-1)^n\left\{-\ln(\varrho_>^2)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\varrho_<}{\varrho_>}\right)^m \cos\left[m(\phi-\frac{n\pi}{2})\right]\right\} \\ &= \frac{\lambda}{2\pi\epsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\varrho_<}{\varrho_>}\right)^m\sum_{n=0}^{3}(-1)^n\cos\left[m(\phi-\frac{n\pi}{2})\right]\end{aligned}$$

Since the ln term cancels after summing over n
Expanding the second sum yields:

$$\begin{aligned}&\sum_{n=0}^{3}(-1)^n\cos\left[m(\phi-\frac{n\pi}{2})\right] \\ &\qquad= \cos(m\phi)-\cos\left(m\phi-\frac{m\pi}{2}\right)+\cos\left(m\phi-m\pi\right)-\cos\left(m\phi-\frac{3m\pi}{2}\right) \\ &\qquad= \cos(m\phi)-\cos(m\phi)\cos\left(\frac{m\pi}{2}\right)+\sin(m\phi)\sin\left(\frac{m\pi}{2}\right)+\cdots\end{aligned}$$

The first two terms show that m must be a multiple of 2, and the second two show that it must be an odd multiple of 2 (otherwise the sum is 0):

$$\sum_{n=0}^{3}(-1)^n\cos\left[m(\phi-\frac{n\pi}{2})\right] = 2\cos(m\phi)$$
$$\text{for } m = 2(2k+1) = 4k+2$$

$$\Phi(\varrho,\phi) = \frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{\varrho_<}{\varrho_>}\right)^{4k+2}\cos\left[(4k+2)\phi\right]$$

(b) Relate the solution of part a to the real part of the complex function

$$w(z) = \frac{2\lambda}{4\pi\epsilon_0}\ln\left[\frac{(z-ia)(z+ia)}{(z-a)(z+a)}\right]$$

where $z = x+iy = \varrho e^{i\phi}$. Comment on the connection to Problem 2.3

Solution.

$$\begin{aligned}\Phi(\varrho,\phi) &= \frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{\varrho_<}{\varrho_>}\right)^{4k+2}\cos\left[(4k+2)\phi\right] \\ &= \Re\left\{\frac{2\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{4k+2}\left(\frac{\varrho_<}{\varrho_>}e^{i\phi}\right)^{4k+2}\right\}\end{aligned}$$

Since

$$\sum_{n\text{ odd}}\frac{Z^n}{n} = \frac{1}{2}\ln\left(\frac{1+Z}{1-Z}\right)$$

$$\begin{aligned}\Rightarrow \Phi(\varrho,\phi) &= \frac{\lambda}{2\pi\epsilon_0}\Re\left\{\ln\left[\frac{1+\left(\frac{\varrho_>}{\varrho_<}e^{i\phi}\right)^2}{1-\left(\frac{\varrho_>}{\varrho_<}e^{i\phi}\right)^2}\right]\right\} \\ &= \frac{\lambda}{2\pi\epsilon_0}\Re\left\{\ln\left[\frac{\left(1+i\frac{\varrho_>}{\varrho_<}e^{i\phi}\right)\left(1-i\frac{\varrho_>}{\varrho_<}e^{i\phi}\right)}{\left(1+\frac{\varrho_>}{\varrho_<}e^{i\phi}\right)\left(1-\frac{\varrho_>}{\varrho_<}e^{i\phi}\right)}\right]\right\}\end{aligned}$$

The interior solution has $\varrho_< = \varrho$ and $\varrho_> = a$ so the solution becomes

$$\begin{aligned}\Phi(\varrho,\phi) &= \frac{\lambda}{2\pi\epsilon_0}\Re\left\{\ln\left[\frac{\left(\varrho+iae^{i\phi}\right)\left(\varrho-iae^{i\phi}\right)}{\left(\varrho+ae^{i\phi}\right)\left(\varrho-ae^{i\phi}\right)}\right]\right\} \\ &= \Re\left[w(\varrho)\right]\end{aligned}$$

The exterior solution has $\varrho_> = \varrho$ and $\varrho_< = a$ so the solution becomes

$$\Phi(\varrho,\phi) = \frac{\lambda}{2\pi\epsilon_0}\Re\left\{\ln\left[\frac{\left(i\varrho+ae^{i\phi}\right)\left(i\varrho-ae^{i\phi}\right)}{\left(\varrho+ae^{i\phi}\right)\left(\varrho-ae^{i\phi}\right)}\right]\right\}$$

Multiply the fraction by i^2 to obtain $\Phi = \Re\left[w(\varrho)\right]$
This is related to problem 2.3 since that problem can be solved with the original line charge and 3 image charges, corresponding to the 4 line charges surrounding the accelerator. Simply take one of the line charges to be at (x_0, y_0) where $x_0 = y_0$

(c) Find expressions for the Cartesian components of the electric field near the origin, expressed in terms of x and y. Keep the $k = 0$ and $k = 1$ terms in the expansion. For $y = 0$ what is the relative magnitude of the $k = 1$ (2^6-pole) contribution to E_x compared to the $k = 0$ (2^2-pole or quadrupole) term?

Solution. The Cartesian components of the electric field are given by

$$E_x = -\cos\theta \frac{\partial \Phi}{\partial \varrho} + \frac{\sin\theta}{\varrho}\frac{\partial \Phi}{\partial \phi}$$
$$E_y = -\sin\theta \frac{\partial \Phi}{\partial \varrho} - \frac{\cos\theta}{\varrho}\frac{\partial \Phi}{\partial \phi}$$

For $\varrho < a$ we have

$$\frac{\partial \Phi}{\partial \varrho} = \frac{2\lambda}{a\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\varrho}{a}\right)^{4k+1}\cos\left[\phi(4k+2)\right]$$
$$\frac{1}{\varrho}\frac{\partial \Phi}{\partial \phi} = -\frac{2\lambda}{a\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\varrho}{a}\right)^{4k+1}\sin\left[\phi(4k+2)\right]$$

Substituting into the original expressions for the components of E:

$$E_x = -\frac{2\lambda}{a\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\varrho}{a}\right)^{4k+1}\left\{\cos\left[\phi\left(4k+2\right)\right]\cos\phi + \sin\left[\phi\left(4k+2\right)\right]\sin\phi\right\}$$
$$= \frac{2\lambda}{a\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\varrho}{a}\right)^{4k+1}\cos\left[\phi\left(4k+1\right)\right]$$
$$E_y = -\frac{2\lambda}{a\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\varrho}{a}\right)^{4k+1}\left\{-\sin\phi\cos\left[\phi\left(4k+2\right)\right] - \cos\phi\sin\left[\phi\left(4k+2\right)\right]\right\}$$
$$= \frac{2\lambda}{a\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\varrho}{a}\right)^{4k+1}\sin\left[\phi\left(4k+3\right)\right]$$

Up to k = 1, this yields:

$$E_x = \frac{2\lambda}{a\pi\epsilon_0}\left[\frac{\varrho}{a}\cos\phi + \left(\frac{\varrho}{a}\right)^5\cos 5\phi\right]$$
$$E_y = \frac{2\lambda}{a\pi\epsilon_0}\left[\frac{\varrho}{a}\sin 3\phi + \left(\frac{\varrho}{a}\right)^5\sin 7\phi\right]$$

For $y = 0$, $\phi = 0, \pi$:

$$E_x = \pm \frac{2\lambda}{a\pi\epsilon_0}\left[\frac{\varrho}{a} \pm \left(\frac{\varrho}{a}\right)^5\right]$$

The relative strength of the $k = 0$ and $k = 1$ terms is ϱ^4