An Illustrated Proof of Front-Door Adjustment Theorem

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Outline

Back-Door Criterion

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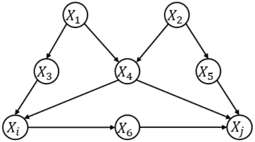

Figure: $S_1 = \{X_3, X_4\}$ and $S_2 = \{X_4, X_5\}$ would qualify under the back-door criterion, but $S_3 = \{X_4\}$ would not because X_4 does not d-separate X_i from X_j along the path $(X_i, X_3, X_1, X_4, X_2, X_5, X_j)$.

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$$P(y|\hat{x}) = \sum_{z} P(y|x, z)P(z). \tag{1}$$

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1. The Back-Door Criterion
   - Definition 1. (Back-Door)
   - Theorem 1. (Back-Door Adjustment)

2. The Front-Door Criterion
   - Definition 2. (Front-Door)
   - Theorem 2. (Front-Door Adjustment)

3. *do* Calculus

4. Proof of Theorem 2.

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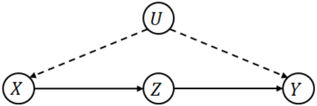

Figure: A diagram representing the front-door criterion.

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$$P(y|\hat{x}) = \sum_{z} P(z|x) \sum_{x'} P(y|x', z) P(x').$$

Rules of *do* Calculus

Preliminary Notation

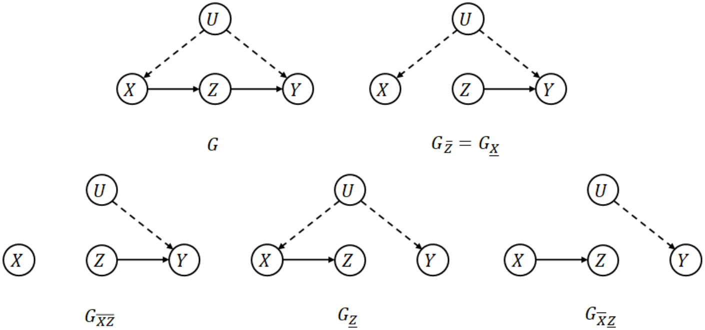

Figure: Subgraphs of G used in the derivation of causal effects.

Inference Rules

Rules of *do* Calculus

Rule 1 (Insertion/deletion of observations):

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where $Z(W)$ is the set of Z-nodes that are not ancestors of any W-node in $G_{\overline{X}}$.

Proof of Front-Door Adjustment Theorem

Step 1: Compute $P(z|\hat{x})$

- $X \perp\!\!\!\perp Z$ in $G_{\underline{X}}$ because there is no outgoing edge from X in $G_{\underline{X}}$, and also by condition (ii) of the definition of the front-door criterion, all back-door paths from X to Z are blocked.

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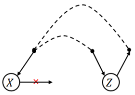

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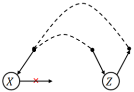

- G satisfies the applicability condition for Rule 2:

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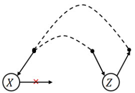

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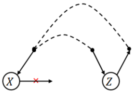

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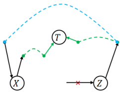

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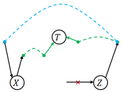

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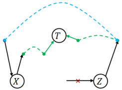

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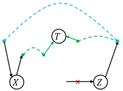

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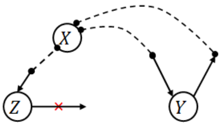

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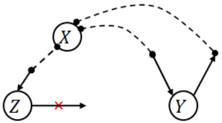

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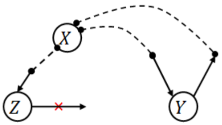

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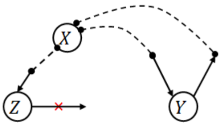

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- This formula is a special case of the back-door formula in Theorem 1.

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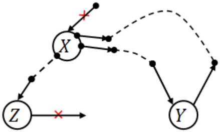

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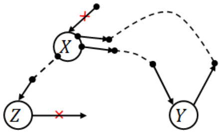

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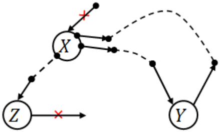

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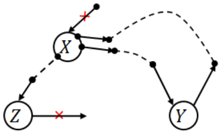

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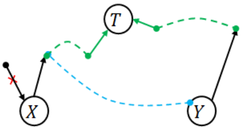

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Step 3 (continued): Compute $P(y|\hat{x})$

- G satisfies the applicability condition for Rule 3:

$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp\!\!\!\perp Z|X, W)_{G_{\overline{X}, \overline{Z(W)}}}.$$

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- In Rule 3, set $y = y, x = z, z = x, w = \emptyset$:

Proof of Front-Door Adjustment Theorem

Step 3 (continued): Compute $P(y|\hat{x})$

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- In Rule 3, set $y = y, x = z, z = x, w = \phi$:

$$P(y|\hat{z}, \hat{x}) = P(y|\hat{z}) \quad \text{because} \quad (Y \perp\!\!\!\perp Z|X)_{G_{\overline{XZ}}}.$$

Proof of Front-Door Adjustment Theorem

Step 3 (continued): Compute $P(y|\hat{x})$

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- $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x}) = \sum_z P(z|x) \sum_{x'} P(y|x', z)P(x')$.

Reference For Further Reading

J. Pearl.
Causality. Models, reasoning, and inference.
Cambridge University Press, 2009.