An Overview of The Back-Door and Front-Door Criteria

A Presentation Based On Sections 3.3 and 3.4 of Pearl's Causality

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Outline

Back-Door Criterion

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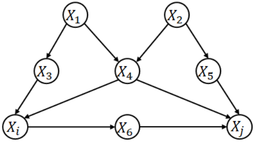

Figure: $S_1 = \{X_3, X_4\}$ and $S_2 = \{X_4, X_5\}$ would qualify under the back-door criterion, but $S_3 = \{X_4\}$ would not because X_4 does not d-separate X_i from X_j along the path $(X_i, X_3, X_1, X_4, X_2, X_5, X_j)$.

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$$P(y|\hat{x}) = \sum_{z} P(y|x, z)P(z). \tag{1}$$

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- The effect of an atomic intervention $do(X_i = x_i')$ is encoded by adding to G a link $F_i \rightarrow X_i$ (see the following Figure), where F_i is a new variable taking values in $\{do(x_i'), idle\}$, x_i' ranges over the domain of X_i, and *idle* represents *no intervention*. Thus, the new parent set of X_i in the augmented network is $PA_i' = PA_i \cup \{F_i\}$, and it is related to X_i by the conditional probability:

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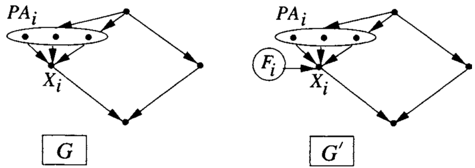

Figure: Representing external intervention F_i by an augmented network G'.

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- $$P(x_i|pa_i') = \begin{cases} P(x_i|pa_i) & \text{if } F_i = \text{idle} \\ 0 & \text{if } F_i = do(x_i') \text{ and } x_i \neq x_i' \\ 1 & \text{if } F_i = do(x_i') \text{ and } x_i = x_i' \end{cases} \quad (2)$$

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$$P(y|\hat{x}) = P'(y|F_x) = \sum_z P'(y|z, F_x)P'(z|F_x) = \sum_z P'(y|z, x, F_x)P'(z|F_x).$$

- According to the condition (i) of the definition of the back-door criterion, no node in Z is a descendant of x. So, all paths between F_x and each node $z \in Z$ have (at least) a collider i.e., $F_x \perp\!\!\!\perp z$ (see the following figure).

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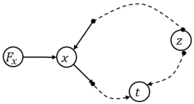

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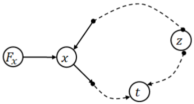

- Therfore, $P'(z|F_x) = P'(z) = P(z)$.

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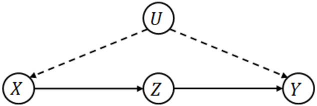

Figure: A diagram representing the front-door criterion.

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1. The Back-Door Criterion
   - Definition 1. (Back-Door)
   - Theorem 1. (Back-Door Adjustment)
   - Proof of Theorem 1.
2. The Front-Door Criterion
   - Definition 2. (Front-Door)
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3. *do* Calculus
4. Symbolic Derivation
5. Example

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If a set of variables Z satisfies the front-door criterion relative to (X, Y) and if $P(x, z) > 0$, then the causal effect of X on Y is identifiable and is given by the formula

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$$P(y|\hat{x}) = \sum_{z} P(z|x) \sum_{x'} P(y|x', z)P(x'). \tag{5}$$

Rules of *do* Calculus

Preliminary Notation

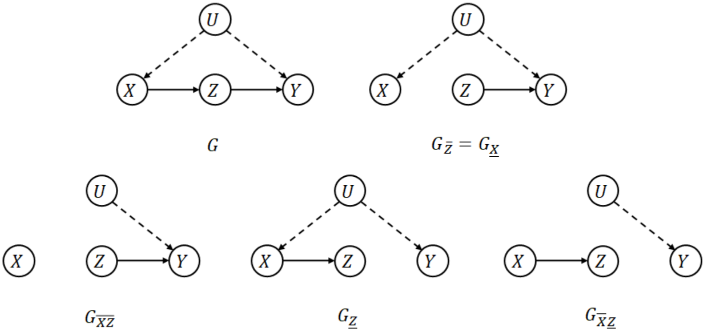

Figure: Subgraphs of G used in the derivation of causal effects.

Inference Rules

Rules of *do* Calculus

Rule 1 (Insertion/deletion of observations):

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Rule 3 (Insertion/deletion of actions):

$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp\!\!\!\perp Z|X, W)_{G_{\overline{X}, \overline{Z(W)}}}.$$

where $Z(W)$ is the set of Z-nodes that are not ancestors of any W-node in $G_{\overline{X}}$.

Symbolic Derivation of Causal Effects: An Example
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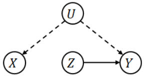

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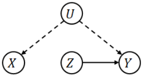

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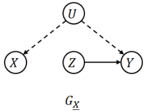

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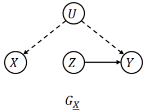

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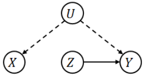

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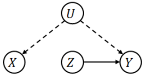

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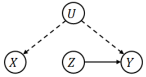

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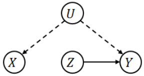

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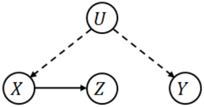

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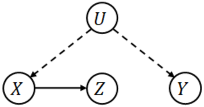

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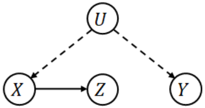

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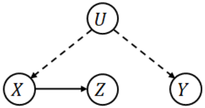
Symbolic Derivation of Causal Effects: An Example
Step 2 (continued): Compute $P(y|\hat{z})$

- $(Z \perp\!\!\!\perp Y|X)_{G_{\underline{Z}}}$ because the path from Z to Y is blocked by X.



$G_{\underline{Z}}$

- G satisfies the applicability condition for Rule 2:
  $P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp\!\!\!\perp Z|X, W)_{G_{\overline{X}\underline{Z}}}$.
- In Rule 2, set $y = y, x = \emptyset, z = z, w = x$:

$$P(y|x, \hat{z}) = P(y|x, z) \quad \text{because} \quad (Z \perp\!\!\!\perp Y|X)_{G_{\underline{Z}}}.$$

Symbolic Derivation of Causal Effects: An Example

Step 2 (continued): Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z}) = \sum_x P(y|x, z)P(x)$.

Symbolic Derivation of Causal Effects: An Example

Step 2 (continued): Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x,\hat{z})P(x|\hat{z}) = \sum_x P(y|x,z)P(x)$.
- This formula is a special case of the back-door formula in Theorem 1.

Symbolic Derivation of Causal Effects: An Example

Step 3: Compute $P(y|\hat{x})$

- $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x})$.

Symbolic Derivation of Causal Effects: An Example

Step 3: Compute $P(y|\hat{x})$

- $P(y|\hat{x}) = \sum_z P(y|z,\hat{x})P(z|\hat{x})$.
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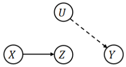

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Symbolic Derivation of Causal Effects: An Example
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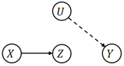

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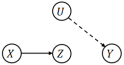

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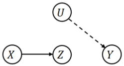

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Symbolic Derivation of Causal Effects: An Example
Step 3 (continued): Compute $P(y|\hat{x})$

- $(Y \perp\!\!\!\perp X|Z)_{G_{\overline{XZ}}}$ because there is no incoming edge to X and no outgoing edge from X in $G_{\overline{XZ}}$.

Symbolic Derivation of Causal Effects: An Example

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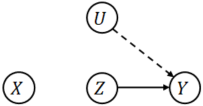

$G_{\overline{XZ}}$

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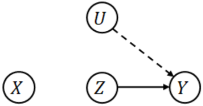

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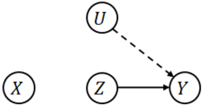

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Symbolic Derivation of Causal Effects: An Example
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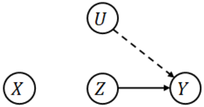

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Step 3 (continued): Compute $P(y|\hat{x})$

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Example: Smoking and the Genotype Theory

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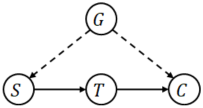

Figure: A diagram representing the story of smoking and the genotype (X=S=Smoking, Z=T=Tar, Y=C=Cancer, and U=G=Genotype (unobserved)).

Example: Smoking and the Genotype Theory

continued

	Group Type	$P(x, z)$ Group Size (% of Population)	$P(Y = 1 \mid x, z)$ % of Cancer Cases in Group
$X = 0, Z = 0$	Nonsmokers, No tar	47.5	10
$X = 1, Z = 0$	Smokers, No tar	2.5	90
$X = 0, Z = 1$	Nonsmokers, Tar	2.5	5
$X = 1, Z = 1$	Smokers, Tar	47.5	85

A hypothetical data set: 95% of smokers and 5% of nonsmokers have developed high levels of tar in their lungs. Moreover, 81% of subjects with tar deposits have developed lung cancer, compared to only 14% among those with no tar deposits.

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A hypothetical data set: 95% of smokers and 5% of nonsmokers have developed high levels of tar in their lungs. Moreover, 81% of subjects with tar deposits have developed lung cancer, compared to only 14% among those with no tar deposits.

- These results seem to prove that smoking is a major contributor to lung cancer. However, the tobacco industry might argue that the table tells a different story that smoking actually decreases ones risk of lung cancer. Their argument goes as follows. If you decide to smoke, then your chances of building up tar deposits are 95%, compared to 5% if you decide not to smoke. In order to evaluate the effect of tar deposits, we look separately at two groups, smokers and nonsmokers. The table shows that tar deposits have a protective effect in both groups: in smokers, tar deposits lower cancer rates from 90% to 85%; in nonsmokers, they lower cancer rates from 10% to 5%. Thus, regardless of whether I have a natural craving for nicotine, I should be seeking the protective effect of tar deposits in my lungs, and smoking offers a very effective means of acquiring those deposits.

Example: Smoking and the Genotype Theory

continued

To settle the dispute between the two interpretations, we now apply the front-door formula (eq. (5)) to the data in the Table of the previous slide:

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To settle the dispute between the two interpretations, we now apply the front-door formula (eq. (5)) to the data in the Table of the previous slide:

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$$\begin{aligned}
P(Y = 1 \,|\, do(X = 1)) &= .05(.10 \times .50 + .90 \times .50) \\
&\quad + .95(.05 \times .50 + .85 \times .50) \\
&= .05 \times .50 + .95 \times .45 = .4525, \\
P(Y = 1 \,|\, do(X = 0)) &= .95(.10 \times .50 + .90 \times .50) \\
&\quad + .05(.05 \times .50 + .85 \times .50) \\
&= .95 \times .50 + .05 \times .45 = .4975.
\end{aligned}$$

Reference For Further Reading

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Causality. Models, reasoning, and inference.
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J. Pearl.
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Statistical Science, 8(3):266–269, 1993.