781 - 2013-03-28

Note Title 2013-03-28

Schöning ex. 56

Show that $F = (\exists x\, P(x) \to P(y))$ is equivalent to

$G = \forall x\, (P(x) \to P(y))$.

is_mother(x) ← mother(x, y)

for all x, x is a mother if there exists ~~someone~~ (y) that x is a mother of.

for all x, x is a mother of for some y, x is the mother of y.

↳ for all x and all y, if x is a mother of y, then

x is a mother.

Let $\mathfrak{A} = (U_{\mathfrak{A}}, I_{\mathfrak{A}})$ be a structure (interpretation) suitable for F and G. Then, we have:

$\mathfrak{A}(\exists x\, P(x) \to P(y)) = 1$

iff $\mathfrak{A}(\sim \exists x P(x) \vee P(y)) = 1$ (defn. of implication)

iff either $\mathfrak{A}(\sim \exists x\, P(x)) = 1$ or $\mathfrak{A}(P(y)) = 1$ (4 on p. 47) iff either $Q(\forall x \sim P(x)) = 1$ or $Q(P(y)) = 1$ (1b on p. 52)

iff $Q(\forall x(\sim P(x) \vee P(y))) = 1$ (2b, with $F = \sim P(x)$ and $G = P(y)$)

iff $Q(\forall x(P(x) \to P(y))) = 1$ (defn of implication)

(One could avoid using the "normal form theorem, 1b 2b, and instead directly use the semantics of predicate calculus, with a few extra steps).

Exercise 57 (Schöning)

Prove that $\forall x \exists y\, P(x,y)$ is a consequence of $\exists u \forall v\, P(v,u)$, but not vice versa.

For the "not vice versa" part, here is a model $(\mathcal{Q})$ of $\forall x\, \exists y\, P(x,y)$ that is not a model of $\exists u \forall v\, P(v,u)$:

$U_{\mathcal{Q}} = \{a,b\}$ $\qquad P^{\mathcal{Q}} \{(a,b),(b,a)\}$

(i) $\mathcal{Q}(\forall x \exists y\, P(x,y)) = 1$ holds iff for each $u \in U_{\mathcal{Q}}$, there is a $v \in U_a$, s.t. $P^a(u,v)$.

(2) Moreover, $Q(\exists y \forall x P(x,y)) = 1$ iff there is a $v \in U_a$, s.t. for all $u \in U_a$, $P^a(u,v)$ holds.

Statement (1) clearly follows from (2).

Exercise 59 (Schöning). Find an equivalent rectified formula for $F = \forall x \exists y P(x, f(y)) \wedge \forall y (Q(x,y) \vee R(x))$
$\wedge \forall z (Q(x,z) \vee R(x))$

$$\forall u\, \exists w\, P(u, f(w)) \wedge \forall z\, (Q(x,z) \vee R(x))$$

$$\forall u\, \exists w\, P(u, f(w)) \wedge (\forall z\, Q(x,z) \vee R(x))$$

$$\forall u\, \exists w\, (P(u, f(w)) \wedge \forall z\, Q(x,z)) \vee$$

$$\forall u\, \exists w\, (P(u, f(w)) \wedge P(x))$$

more work to obtain

a prenex (and rectified world)