Exercise 5.3.1: Find a satisfiable formula \( F \) of predicate logic with identity such that for every model \( A \) of \( F \), \( |\mathcal{U}_A| \leq 2 \).

This exercise seems to contradict the previous exercises. Convince yourself that there is no contradiction!

\[
F = \forall x \forall y \forall z \left( (x = y) \lor (y = z) \lor (x = z) \right)
\]

(At least two of \( x, y, z \) must be equal.)

There is no contradiction, b/c we use predicate logic.
with equality here; we did not do exercise 49.

A similar example (yes, here, Ex. 12.2). Here is a formula of pred. logic w/equality that has only models whose universe has cardinality 3:

\[ \exists x \exists y \exists z \forall u \left( (x \neq y \land y \neq z \land x \neq z) \land (x = u \lor y = u \lor z = u) \right) \]

at least three individuals at most three individuals

\[ t_1 \neq t_2 \text{ is an abbreviation of } \neg (t_1 = t_2) \]
individual means member of the universe (aka domain of discourse), aka ground set)
Exercise 52: Find formulas of predicate logic with identity (cf. Exercise 46) which contain a binary predicate symbol $P$ (or a unary function symbol $f$) and which express:

(a) $P$ is a anti-symmetric relation.

(b) $f$ is a one-one function.

(c) $f$ is a function which is onto.

\[ (a) \ \forall x \forall y (P(x, y) \land P(y, x)) \quad (P \text{ is anti-symmetric}) \]

\[ (b) \ \forall x \forall y \ (f(x) = f(y) \rightarrow (x = y)) \quad (f \text{ is one-to-one or injective}) \]
(c) \( \forall y \exists x \ (f(x) = y) \) \ (f is onto or surjective).

Every element of the range (co-domain) \( y \) of \( f \) is the image of some element of the domain \( x \) of \( f \).

Note: a function that is both one-to-one and onto is called a one-to-one correspondence (aka bijection).
Exercise 53: Formulate a satisfiable formula $F$ in predicate logic with identity (cf. Exercise 46) in which a binary function symbol $f$ occurs such that for every model $\mathcal{A}$ of $F$ it holds:

$$(\mathcal{U}_A, f^A) \text{ is a group.}$$

(Recall: the integers and plus form a group.)

$$F = \forall x \forall y \forall z \left[ f(x, f(y, z)) = f(f(x, y), z) \right] \quad \text{(associativity)}$$

$$\land \exists x \left[ \forall y \left( f(x, y) = y \right) \right] \quad \text{(neutral element)}$$
\[ \forall y \exists z (f(y, z) = x) \] (inverse)

It may be better to define the neutral element explicitly, by \( \forall y \ f(e, y) = y \)
Exercise 54: A stack is a well known abstract data structure in Computer Science. Certain predicates and functions (better: operations) are defined to test the status of the stack or to manipulate the stack. E.g., $\text{IsEmpty}$ is a unary predicate expressing the fact that the stack is empty, and $\text{nullstack}$ is a constant that stands for the empty stack. Further, $\text{top}$ (giving the top element of the stack) and $\text{pop}$ are unary functions, and $\text{push}$ is a binary function (which gives the new stack after pushing a new element on top of the given stack).

"Axiomatize" these operations which are allowed on a stack by a formula in predicate logic in such a way that every model of this formula can be understood as an (abstract) stack.

Hint: A possible part of such a formula might be the formula

$$\forall x \forall y (\text{top}(\text{push}(x, y)) = x)$$
\( F = \text{Is Empty (null stack)} \)

\( \land \forall x \forall y \neg \text{Is Empty (push (x, y))} \)

\( \land \forall x \forall y \ (\text{top (push (x, y))} = x) \)

\( \land \forall x \forall y \ (\text{pop (push (x, y))} = y) \)

\( \land \forall x \ (\neg \text{Is Empty (x)} \rightarrow \ (\text{push (top (x), pop (x))} = x) ) \)
It is even more interesting to observe which pairs of very similar looking formulas are not equivalent:

\[
\begin{align*}
(1) & \quad (\forall x F \lor \forall x G) \not\equiv \forall x (F \lor G) \\
(2) & \quad (\exists x F \land \exists x G) \not\equiv \exists x (F \land G)
\end{align*}
\]

Exercise 55: Confirm this by exhibiting counterexamples (i.e. structures which are models for one of the formulas, but not for the other).

For (1), consider the following structure \( \mathcal{A} \); \( U_\mathcal{A} = \{a, b\} \)

and \( \mathcal{A}(H) = \{a\} \), \( \mathcal{A}(J) = \{b\} \)

\[ F = H(x) \land J(x), \text{ so (i) becomes } \forall x H(x) \lor \forall x J(x) \not\equiv \forall x (H(x) \lor J(x)) \]
So, \( A(\forall x H(x) \lor \forall x J(x)) = 0 \), but
\[ A(\forall x (H(x) \lor J(x))) = 1 \text{, so we showed (1)} \]

For (2) note that
\[ A(\exists x H(x) \land \exists x J(x)) = 1 \text{, but} \]
\[ A(\exists x (H(x) \lor J(x))) = 0 \].

Do Exercises 56 + 57 [Sch"oning]: HW 3
due Thursday one week from today.