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Note Title 2013-03-21

Exercise 51: Find a satisfiable formula F of predicate logic with identity such that for every model $\mathcal{A}$ of F, $|U_{\mathcal{A}}| \leq 2$.

This exercise seems to contradict the previous exercise. Convince yourself that there is no contradiction!

$$F = \forall x \forall y \forall z \left((x=y) \vee (y=z) \vee (x=z) \right)$$

(Atleast two of x, y, z must be equal.)

There is no contradiction, b/c we use predicate logic with equality here; we did not do exercise 49.

A similar example (Yasuhara, Ex. 12.2). There is a formula of pred. logic w/ equality that has only models whose universe has cardinality 3;

$$\exists x \exists y \exists z \forall u \Big(\underbrace{(x \neq y \wedge y \neq z \wedge x \neq z)}_{\text{at least three individuals}} \wedge \underbrace{(x = u \vee y = u \vee z = u)}_{\text{at most three individuals}} \Big)$$

$t_1 \neq t_2$ is an abbreviation of $\neg(t_1 = t_2)$

individual means member of the universe (aka domain of discourse), aka ground set)

Exercise 52: Find formulas of predicate logic with identity (cf. Exercis 46) which contain a binary predicate symbol P (or a unary function symbc f) and which express:

(a) P is a anti-symmetric relation.

(b) f is a one-one function.

(c) f is a function which is onto.

(a) $\forall x \forall y \neg (P(x,y) \wedge P(y,x))$ (P is anti-symmetric)

(b) $\forall x \forall y \ (f(x) = f(y) \rightarrow (x = y))$ (f is one-to-one or injective)

(c) $\forall y\, \exists x\, (f(x) = y)$ (f is onto or surjective)

Every element of the range (co-domain) of f is the image of some element of the domain of f.

Note: a function that is both one-to-one and onto is called a one-to-one correspondence (aka bijection).

Exercise 53: Formulate a satisfiable formula F in predicate logic with identity (cf. Exercise 46) in which a binary function symbol f occurs such that for every model $\mathcal{A}$ of F it holds:

$$(U_{\mathcal{A}}, f^{\mathcal{A}}) \text{ is a group.}$$

(Recall: the integers and plus form a group.)

$F = \forall x \forall y \forall z \, [f(x, f(y,z)) = f(f(x,y), z)]$ (associativity)

$\wedge \; \exists x \, [\forall y \, (f(x,y) = y)$ (neutral element)

$$\wedge \forall y \exists z (f(y,z) = x)]$$ (inverse)

It may be better to define the neutral element explicitly, by $\forall y \ f(e,y) = y$

Exercise 54: A *stack* is a well known abstract data structure in Computer Science. Certain predicates and functions (better: operations) are defined to test the status of the stack or to manipulate the stack. E.g., *IsEmpty* is a unary predicate expressing the fact that the stack is empty, and *nullstack* is a constant that stands for the empty stack. Further, *top* (giving the top element of the stack) and *pop* are unary functions, and *push* is a binary function (which gives the new stack after pushing a new element on top of the given stack).

"Axiomatize" these operations which are allowed on a stack by a formula in predicate logic in such a way that every model of this formula can be understood as an (abstract) stack.

Hint: A possible part of such a formula might be the formula

$$\forall x \forall y (top(push(x, y)) = x)$$

$$
\begin{aligned}
F = \ & \mathrm{IsEmpty}(\mathrm{nullstack}) \\
& \wedge \forall x \forall y \ \neg \mathrm{IsEmpty}(\mathrm{push}(x,y)) \\
& \wedge \forall x \forall y \ (\mathrm{top}(\mathrm{push}(x,y)) = x) \\
& \wedge \forall x \forall y \ (\mathrm{pop}(\mathrm{push}(x,y)) = y) \\
& \wedge \forall x \ (\neg \mathrm{IsEmpty}(x) \rightarrow \\
& \qquad (\mathrm{push}(\mathrm{top}(x), \mathrm{pop}(x)) = x))
\end{aligned}
$$

It is even more interesting to observe which pairs of very similar looking formulas are *not* equivalent:

$$
\begin{aligned}
&(1) \quad (\forall x F \vee \forall x G) \not\equiv \forall x (F \vee G)\\
&(2) \quad (\exists x F \wedge \exists x G) \not\equiv \exists x (F \wedge G)
\end{aligned}
$$

Exercise 55: Confirm this by exhibiting counterexamples (i.e. structures which are models for one of the formulas, but not for the other).

For (1), consider the following structure $\mathcal{A}$: $U_{\mathcal{A}} = \{a, b\}$

and $\mathcal{A}(H) = \{(a)\}$, $\mathcal{A}(J) = \{(b)\}$

$F = H(x)$, $G = J(x)$, so (1) becomes $\forall x\, H(x) \vee \forall x\, J(x) \not\equiv \forall x (H(x) \vee J(x))$

So, $\mathfrak{A}(\forall x\, H(x) \vee \forall x\, J(x)) = 0$, but

$\mathfrak{A}(\forall x\,(H(x) \vee J(x))) = 1$, so we showed (1)

For (2); note that

$\mathfrak{A}(\exists x\, H(x) \wedge \exists x\, J(x)) = 1$, but

$\mathfrak{A}(\exists x\,(H(x) \wedge J(x))) = 0$.

Do Exercises 56 + 57 [Schöning]: HW 3

due Thursday one week from today.