Ex. 45 (Schröning)

\[ F_1 = \forall x P(x, x) \]
\[ F_2 = \forall x \forall y (P(x, y) \rightarrow P(y, x)) \]
\[ F_3 = \forall x \forall y \forall z ((P(x, y) \land P(y, z)) \rightarrow P(x, z)) \]

In all interpretations, we present have the same universe,
\[ U = \{ 1, 2, 3 \} \] (domain of discourse; universe)

\( \mathcal{A} \) is a structure (interpretation) for \( F_2 \) and \( F_3 \), but not a model.
for $F_1: \quad P^\alpha = \emptyset$ \quad (This \ interpretation \ maps \ P
d to \ the \ empty \ relation.)

$F_1^\alpha = \text{false} (= 0), \quad F_2^\alpha = F_3^\alpha = \text{true} (= 1)$

$A_2$ is a model for $F_1$ and $F_3$, but not for $F_2$:

$P^\alpha_2 = \{ (1,1), (2,2), (3,3), (1,2) \}$

$A_3$ is a model for $F_1$ and $F_2$, but not for $F_3$:

$\{ (1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (2,2) \}$
Ex. 46 (Predicate logic with equality)

Exercise 46: In predicate logic with identity the symbol $=$ is also permitted in formulas (as a special binary predicate with a fixed interpretation) which is to be interpreted as identity (of values) between terms. How has the syntax (i.e. the definition of formulas) and the semantics (the definition of $A(F)$) of predicate logic to be extended to obtain the predicate logic with identity?

Syntax: if $t_1$ and $t_2$ are terms, then $t_1 = t_2$ is a formula.

Semantics: if $F$ has the form $t_1 = t_2$, then

$$A(F) = \begin{cases} 1 & \text{if } A(t_1) = A(t_2) \\ 0 & \text{otherwise} \end{cases}$$
Exercise 47: Which of the following structures are models for the formula 

\[ F = \exists x \exists y \exists z (P(x, y) \land P(z, y) \land P(x, z) \land \neg P(z, x)) \] ?

(a) \( U_A = \mathbb{N}, P^A = \{(m, n) \mid m, n \in \mathbb{N}, m < n\} \)

(b) \( U_A = \mathbb{N}, P^A = \{(m, m + 1) \mid m \in \mathbb{N}\} \)

(c) \( U_A = 2^\mathbb{N} \) (the power set of \( \mathbb{N} \)),

\[ P^A = \{(A, B) \mid A, B \subseteq \mathbb{N}, A \subseteq B\} \]

(a) asks, in effect, whether there is a solution to the following system of inequalities:

\[
\begin{align*}
\exists x < y & \\
\exists z < y & \\
\exists x < z & \quad x = 1 \\
(2 \geq z) & \quad z = 2
\end{align*}
\]

\[ \checkmark \]

It is a solution.
(b) Similarly, is there a solution to the following set of equations?
\[
\begin{cases}
  x = y + 1 \\
  z = y + 1 \\
  x = 2 + 1 \\
  (z \neq x + 1)
\end{cases}
\]

There is no solution.

\[ P^a = \{(0,1), (1,2), (2,3), (3,4), \ldots \} \]

(The successor relation)

\[ U^n = 2^\mathbb{N} = \{\{0\}, \{0,1\}, \{0,1,2\}, \ldots \} \]

\[ P^a \text{ the (non-proper) subset relation} \]
Exercise 48: Let $F$ be a formula, and let $x_1, \ldots, x_n$ be the variables that occur free in $F$. Show:

(a) $F$ is valid if and only if $\forall x_1 \forall x_2 \ldots \forall x_n F$ is valid,

(b) $F$ is satisfiable if and only if $\exists x_1 \exists x_2 \ldots \exists x_n F$ is satisfiable.

This makes clear what the highlighted
If for a formula $F$ and a suitable structure $A$ we have $A(F) = 1$, then we denote this by $A \models F$ (we say, $F$ is true in $A$, or $A$ is a model for $F$).

Some authors (e.g., Yasu here) call the mapping of variables to elements of the universe assignment. This exercise emphasizes that a formula is valid if it is true for all assignments to the free variables.
Exercise 49: Find a closed satisfiable formula $F$, such that for every model $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ of $F$, $|U_\mathcal{A}| \geq 3$.

$$\forall x \in (x, x) \land \forall x \forall y \forall z \left[ \neg F(x, y) \land \neg F(x, z) \land \neg F(y, z) \right]$$

Exercise 50: Let $F$ be a satisfiable formula and let $\mathcal{A}$ be a model for $F$ with $|U_\mathcal{A}| = n$. Show that for every $m \geq n$ there is a model $\mathcal{B}_m$ for $F$ with $|U_{\mathcal{B}_m}| = m$. Furthermore, there is a model $\mathcal{B}_\infty$ for $F$ with $|U_{\mathcal{B}_\infty}| = \infty$.

Hint: Pick some element $u$ from $U_\mathcal{A}$, and add new elements to $U_{\mathcal{B}_m}$ having the same properties as $u$. Let $(U_\mathcal{B}, I_\mathcal{B})$ be a model. Define $\mathcal{B} = \mathcal{B}_m$ in this way:

$$U_{\mathcal{B}_m} = U_\mathcal{A} \cup \{ b_i, \ldots, b_{m-n} \} \quad \text{(so that } |U_{\mathcal{B}_m}| = m)$$
We widen $I_a$ to $I_b$. $b_i$, $b_j$.

If $(\ldots a, \ldots, a, \ldots) \in P^a$, then $(\ldots b_i, \ldots, b_j, \ldots) \in P^B$.

Similarly, for function names

$f^B ((\ldots, b_i, \ldots, b_j, \ldots)) = f^A ((\ldots, a, \ldots, a, \ldots))$

$B([F]) = \bigcup \{ [F] \circ [x/u] \}$ for all variables $x$, all formulas $F$ and $u \in \{ b_1, \ldots, b_{m-n} \}$. 