Question 1

(10 points)
Prove the converse of the deduction theorem: If $B_1, \ldots, B_{k-1} \vdash (B_k \supset C)$, then $B_1, \ldots, B_{k-1}, B_k \vdash C$.

Prove: If $\vdash (B_k \supset C)$ then $B_k \vdash C$.

(1) $\vdash (B_k \supset C)$ assumption

(2) $B_k \vdash (B_k \supset C)$ b/c any proof from axioms a6a is also a proof from axioms
(3) \( B_k + B_k \) and hypotheses.

(4) \( B_k + C \) mostly focuses on (2), (3)

["monotonicity of the propositional calculus"]
(30 points) (This is exercise 3 in Schöning.) A formula $G$ is called a (logical) consequence of set of formulas $\{F_1, F_2, \ldots, F_k\}$ if for every assignment $A$ that is suitable for each of $F_1, F_2, \ldots, F_k$ and $G$ it follows that, whenever $A$ is a model for $F_1, F_2, \ldots, F_k$, then it is also a model for $G$. (This is indicated $F_1, F_2, \ldots, F_k \models G$ or $A \models G$.)

Show that the following assertions are equivalent:

1. $G$ is a logical consequence of $F_1, F_2, \ldots, F_k$.
2. $((\land_{i=1}^{k} F_i) \rightarrow G)$ is a tautology.
3. $((\land_{i=1}^{k} F_i) \rightarrow G)$ is unsatisfiable. (Hint: Prove 1 $\rightarrow$ 2, $\neg 3 \rightarrow \neg 2$, and 3 $\rightarrow$ 1.)

(recall: this part was not graded)

\[ \square \quad \text{(error in underline)} \]

\[ Q \]

1 $\rightarrow$ 2, consider a suitable assignment $A$ (i.e., a suitable interpretation) of $F_1, F_2, \ldots, F_k$ that is not a model of $F_1, \ldots, F_k$. Then at least one of $F_1, F_2, \ldots, F_k$, say $F_i$, is false in $A$, i.e., $A(F_i) = 0$. Therefore,
\( a(\bigwedge_{i=1}^{k} F_i) = 0 \). Therefore \( a(\bigwedge_{i=1}^{k} F_i \rightarrow \ell) = 1 \).

Now, consider a suitable assignment \( a \) of \( F_1, F_2, \ldots, F_k \) that is a model of \( F_1, F_2, \ldots, F_k \). Then, \( a(F_i) \) is true for every \( 1 \leq i \leq k \). Therefore \( a(\bigwedge_{i=1}^{k} F_i) = 1 \).

But, since \( \ell \) is a logical consequence of \( F_1, \ldots, F_k \), \( a(\ell) = 1 \). So, \( a(\bigwedge_{i=1}^{k} F_i \rightarrow \ell) = 1 \).
\[73 \rightarrow 72\]

Let \( \alpha \) be a suitable assignment of \( \bigwedge_{i=1}^{k} f_i \rightarrow \neg \alpha \)

that is a model of \( \alpha \), i.e., \( \alpha \left( \bigwedge_{i=1}^{k} f_i \rightarrow \neg \alpha \right) = 1 \).

Then, either (i) \( \alpha \left( \bigwedge_{i=1}^{k} f_i \right) = 0 \), and therefore

\[\alpha \left( \bigwedge_{i=1}^{k} f_i \rightarrow \neg \alpha \right) = 1\]

or

(ii) \( \alpha \left( \bigwedge_{i=1}^{k} f_i \right) = 1 \) and \( \alpha \left( \neg \alpha \right) = 1 \), and therefore

\[\alpha \left( \bigwedge_{i=1}^{k} f_i \rightarrow \neg \alpha \right) = 0\]
2 \implies 3  
Assuming \text{73}  
\text{For every suitable assignment } \alpha \text{ for which}

\[ \alpha \left( \bigwedge_{i=1}^{k} F_i \right) = 1 \]

then \[ \alpha (\neg A) = 1 \]

Then, \[ \alpha (A) = 0 \] Therefore,

\[ \alpha \left( \bigwedge_{i=1}^{k} F_i \rightarrow A \right) = 0 \]

\text{For every suitable assignment } \alpha \text{ for which}

\[ \alpha \left( \bigwedge_{i=1}^{k} F_i \right) = 0 \]

then \[ \alpha \left( \bigwedge_{i=1}^{k} F_i \rightarrow A \right) = 1 \]
73 \to 72

Let $\alpha$ be a suitable assignment of $\bigwedge_{i=1}^{k} f_i \land \neg \psi$ that is a model. Then,

$$\alpha(\bigwedge_{i=1}^{k} f_i \land \neg \psi) = 1,$$

so $\alpha(\bigwedge_{i=1}^{k} f_i) = 1$ and

$$\alpha(\bigwedge_{i=1}^{k} \neg \psi) = 1,$$

so $\alpha(\bigwedge_{i=1}^{k} \neg \psi) = 0$, so

$$\alpha(\bigwedge_{i=1}^{k} f_i \rightarrow \psi) = 0,$$

and therefore

$$\bigwedge_{i=1}^{k} f_i \rightarrow \psi$$

is not a tautology.
3 \rightarrow 1

Assume $\alpha$ is a model of $\bigwedge_{i=1}^{k} F_i$. Then, by (3),
$\alpha(-4) = 0$, so $\alpha(4) = 1$. Therefore, if $\alpha$
 is a model of each of the $F_i$, then
$\alpha$ is a model of $G$.

Question 2(c).

Consider $KB \cup \{ 2g \}$. Run the marking algorithm.
$g$ is not marked. Therefore, in the minimal
model of KB, $q$ could be false. Therefore, there is no model of KB in which $q$ is false. Therefore, $q$ does not logically follow from KB.