Note Title 2013-03-05

## Question 1

(10 points)

Prove the converse of the deduction theorem: If $\mathbf{B_1}, ..., \mathbf{B_{k-1}} \vdash (\mathbf{B_k} \supset \mathbf{C})$, then $\mathbf{B_1}, ..., \mathbf{B_{k-1}}, \mathbf{B_k} \vdash \mathbf{C}$

Prove: If $\vdash (B_k \supset C)$ then $B_k \vdash C$.

(1) $\vdash (B_k \supset C)$ assumption

(2) $B_k \vdash (B_k \supset C)$ b/c any proof from axioms alone is also a proof from axioms

and hypotheses. ["monotonicity of the propositional calculus"]

(3) $B_k \vdash B_k$ hypothesis

(4) $B_k \vdash C$ modus ponens on (2), (3)

(30 points) (This is exercise 3 in Schöning.) A formula $G$ is called a *(logical) consequence* of set of formulas $\{F_1, F_2, ..., F_k\}$ if for every assignment $\mathcal{A}$ that is suitable for each of $F_1, F_2, ..., F_k$ and $G$ it follows that, whenever $\mathcal{A}$ is a model for $F_1, F_2, ..., F_k$, then it is also a model for $G$. (This is indicated $F_1, F_2, ..., F_k \models G$ or $\mathcal{A} \models G$.)

Show that the following assertions are equivalent:

1. $G$ is a logical consequence of $F_1, F_2, ..., F_k$.
2. $((\wedge_{i=1}^{k} F_i) \to G)$ is a tautology.
3. $((\wedge_{i=1}^{k} F_i) \wedge \neg G)$ is unsatisfiable.

(Hint: Prove $1 \to 2$, $\neg 3 \to \neg 2$, and $3 \to 1$.)

(recall): this part was not graded)

error in midterm

$\mathcal{A}$

$1 \to 2$, Consider a suitable assignment $\mathcal{A}$ (i.e., a suitable interpretation) of $F_1, F_2, \ldots, F_k$ that is not a model of $F_1, \ldots, F_k$. Then at least one of $F_1, F_2, \ldots F_k$, say $F_i$, is false in $\mathcal{A}$, i.e., $\mathcal{A}(F_i) = 0$. Therefore,

$a\left(\bigwedge_{i=1}^{k} F_i\right) = 0$. Therefore $\alpha\left(\bigwedge_{i=1}^{k} F_i \to G\right) = 1$.

Now, consider a suitable assignment $\alpha$ of $F_1, F_2, \ldots F_k$ that is a model of $F_1, F_2, \ldots, F_k$. Then, $a(F_i)$ is true for every $1 \leq i \leq k$. Therefore $\alpha\left(\bigwedge_{i=1}^{k} F_i\right) = 1$.

But, since $G$ is a logical consequence of $F_1, \ldots, F_k$, $\alpha(G) = 1$. So, $\alpha\left(\bigwedge_{i=1}^{k} F_i \to G\right) = 1$.

$\neg 3 \to \neg 2$

Let $\mathcal{A}$ be a suitable assignment of $\bigwedge_{i=1}^{k} F_i \to \neg G$ that is a model of it, i.e, $\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i \to \neg G\right) = 1$.

Then, either (i) $\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i\right) = 0$, and therefore

$\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i \to G\right) = 1$, or

(ii) $\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i\right) = 1$ and $\mathcal{A}\left(\neg G\right) = 1$, and

therefore $\mathcal{A}(G) = 0$, and therefore

$\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i \to G\right) = 0$

$2 \to 3$ Assuming $\neg 3$

For every suitable assignment $\mathcal{A}$ for which

$\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i\right) = 1$, then $\mathcal{A}(\neg G) = 1$.

Then, $\mathcal{A}(G) = 0$. Therefore,

$\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i \to G\right) = 0$

For every suitable assignment $\mathcal{A}$ for which

$\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i\right) = 0$, then $\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i \to G\right) = 1$.

$\neg 3 \to \neg 2$

Let $\mathcal{A}$ be a suitable assignment of $\bigwedge_{i=1}^{k} F_i \wedge \neg G$ that is a model. Then,

$\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i \wedge \neg G\right) = 1$, so $\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i\right) = 1$ and

$\mathcal{A}\left(\bigwedge_{i=k}^{k} \neg G\right) = 1$, so $\mathcal{A}\left(\bigwedge_{i=1}^{k} G\right) = 0$, so

$\mathcal{A}\left(\bigwedge_{i=1}^{k} F_i \to G\right) = 0$, and therefore

$\bigwedge_{i=1}^{k} F_i \to G$ is not a tautology.

$3 \to 1$

Assume $\alpha$ is a model of $\bigwedge_{i=1}^{k} F_i$. Then, by (3), $\alpha(\neg G) = 0$, so $\alpha(G) = 1$. Therefore, if $\alpha$ is a model of each of the $F_i$, then $\alpha$ is a model of $G$.

Question 2(c).

Consider $KB \cup \{\neg q\}$. Run the marking algorithm. $q$ is not marked. Therefore, in the minimal model of KB, q could be false. Therefore, there is exists a model of KB in which q is false. Therefore, q does not logically follow from KB