The compactness theorem for the propositional calculus.

Theorem [1.2.3 in Loveland, 1978 (Automated Theorem Proving: A Logical Basis, North-Holland).]

A set $S$ of formulas is satisfiable iff every finite subset $T$ of $S$ is satisfiable.
(This is much more important for the predicate calculus (1st-order logic) than for the propositional calculus.)

Proof. It is immediate that if is a satisfiable set of formulas then any subset of \( S \) is satisfiable.

To show the converse we show that if \( S \) is unsatisfiable,
Then there is a finite subset of \( S \) (name it \( T \)) that is also unsatisfiable.

Assume that \( S \) is unsatisfiable. Then,
\[ t \in (S \cup (A \land \neg A)) \quad \text{b/c it is a tautology} \]
\[ S + A \land \neg A \quad \text{converse of the deduction theorem} \]

But proof are finite, so there is a finite subset of \( S \cup T \), that contains all assumptions used in the proof of
\[(A \land \lnot A), \text{ i.e.} \]

\[T \vdash A \land \lnot A\]

\[\vdash (T \Rightarrow (A \land \lnot A)) \quad \text{Induction Theorem;} \]

\[(A \land \lnot A) \text{ is a contradiction;} \]

\[\vdash (T \Rightarrow (A \land \lnot A)) \quad \text{is a tautology; so} \]

\[T \text{ is unsatisfiable.}\]

You can find much more complicated proofs of this theorem that use König's Lemma.
or levels of 'length' of proofs, but the proof given is simpler.

First-order logic (Ca. 10 [Yasuhara])
The most general
First-order language $L(F)$: The symbols of $L(F)$ are,

1. The individual variables: $x_1, x_2, x_3, \ldots$
2. The predicate symbols: $p_1, p_1^1, p_1^2, \ldots, p_2, p_2^2, \ldots, p_n, p_n^2, \ldots$
3. The function symbols: $f_1, f_1^1, \ldots, f_2, f_2^2, \ldots, f_n, f_n^2, \ldots$
4. Names for constants: \( a_0, a_1, a_2, a_3, \ldots \)

5. Propositional connectives: \( \neg, \land, \lor, \exists, \forall \).

6. Commas and parentheses: \( \langle, \rangle \)

7. The universal quantifier: \( \forall \).

To define the formulas \( F(f) \), we first define certain sets of words on the symbols:

- The terms of \( L(f) \): (1) any individual variable \( a \) is a term, any name of constant is a term, \( \ldots \).
(2) If $t_1, \ldots, t_n$ are terms and $f_i$ is a function symbol, then $f_i^n(t_1, \ldots, t_n)$ is a term.

- The abstract formulas of $L(F)$: if $t_1, \ldots, t_n$ are terms, and $p_i$ is a predicate symbol, then $p_i^n(t_1, \ldots, t_n)$ is an atomic formula.

- The formulas $F(F)$ of $L(F)$: (1) every atomic formula is in $F(F)$. (2) If
A and B are in F(F), then so are
(A \rightarrow B), (A \land B), (A \lor B), (A \equiv B), \neg A,
(3) if A is in F(F) and x_i is an individual variable, then \( \forall x_i : A \) is in \( F(F) \).
All first order languages are subsets of \( L(F) \).

(axiom schema)
The axioms for first-order logic are:
\[(A \supset (B \supset A))\]
\[(((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))))\]
\[(((\neg A \supset \neg B) \supset (B \supset A))\]
\[(\forall x_i) A(x_i) \supset A(t) \text{ if } t \text{ is free for } x_i \text{ in } A\]

\[(\forall x_i) (A \supset B) \supset (A \supset (\forall x_i) B) \text{ if } A \text{ contains no free occurrence of } x_i\]

The rules of inference are:

If \(A\) and \(A \supset B\), then \(B\) - modus ponens
If \(A\) then \((\forall x_i) A\) - universal generalization
The connective $\exists$ is usually added.

**Resolution (Propositional)**

Resolution applies to formulas in Conjunctive Normal Form (CNF), i.e.,

$$F = (L_{11} \lor \ldots \lor L_{1n}) \land \ldots \land (L_{k1} \lor \ldots \lor L_{kn})$$

(This formula has $k$ clauses, each of which is a disjunction of $(n_1, \ldots, n_k)$ literals.)
A literal is either a propositional variable or a negated propositional variable (e.g., \(p_i\), \(\neg p_i\)).

CNF formulas are often represented in set notation, i.e., a CNF formula is a set of clauses, each of which is a set of literals.

**Definition.** Resolvent