781 2013-02-12

Note Title 2013-02-12

The compactness theorem for the propositional calculus

Theorem [1.2.3 in Loveland, 1978 (Automated Theorem Proving: A Logical Basis, North-Holland).]

A set S of formulas is satisfiable iff every finite subset T of S is satisfiable.

(This is much more important for the predicate calculus (1st-order Logic) than for the propositional calculus.)

<u>Proof:</u> It is immediate that if S is a satisfiable set of formulas then any subset of S is satisfiable.

To show the converse we show that if S is unsatisfiable

Then there is a finite subset of S (name it T) that is also unsatisfiable.

Assume that S is unsatisfiable. Then,

$\vdash (S \supset (A \wedge \sim A))$ b/c it is a tautology

$S \vdash A \wedge \sim A$ converse of the deduction theorem

But proofs are finite, so there is a finite subset of S, T, that contains all assumptions used in the proof of

$(A \wedge \sim A)$, i.e.

$T \vdash A \,\&\, \sim A$

$\vdash (T \supset (A \,\&\, \sim A))$ deduction theorem;

$(A \,\&\, \sim A)$ is a contradiction,

$(T \supset (A \,\&\, \sim A))$ is a tautology, so

T is unsatisfiable.

$\square$

You can find much more complicated proofs of this theorem that use König's Lemma, or "levels" of "lengths" of proofs, but the proof given is simpler.

First-order logic (Ch. 10 [Yasuhara])

The most general first-order language $L(F)$, The symbols of $L(F)$ are,

1. The individual variables: $x_1, x_2, x_3, \ldots$
2. The predicate symbols: $P_1^0, \ldots P_n^0, \ldots, P_1^1, P_2^1, \ldots P_1^2, P_2^2, \ldots, P_1^4, P_2^4, \ldots$
3. The function symbols: $f_1^1, f_2^1, \ldots\ f_1^2, f_2^2, \ldots, f_1^4, f_2^4, \ldots$

4. Names for constants: $a_0, a_1, a_2, a_3, \ldots$

5. Propositional connectives: $\supset, \wedge, \vee, \equiv, \sim$.

6. Comma and parentheses: , ( )

7. The universal quantifier: $\forall$.

To define the formulas $F(F)$, we first define certain sets of words on the symbols:

- The <u>terms</u> of $L(F)$: (i) any individual variable is a term, any name of constant is a term;

(2) if $t_1, \ldots, t_n$ are terms and f_i^n is a function symbol, then $f_i^n(t_1, \ldots, t_n)$ is a term

- The atomic formulas of $L(F)$: if $t_1, \ldots, t_n$ are terms, and P_i^n is a predicate symbol, then $P_i^n(t_1, \ldots, t_n)$ is an atomic formula.

The formulas $F(F)$ of $L(F)$; (1) every atomic formula is in $F(F)$. (2) If

A and B are in $F(F)$, then so are

$(A \supset B)$, $(A \wedge B)$, $(A \vee B)$, $(A \equiv B)$, $\sim A$.

(3) if A is in $F(F)$ and x_i is an individual variable, then $(\forall x_i) A$ is in $F(F)$.

All first order languages are subsets of $L(F)$.

The axioms (axiom schemata) for first-order logic are:

$$(A \supset (B \supset A))$$

$$((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))$$

$$((\sim A \supset \sim B) \supset (B \supset A))$$

$(\forall x_i) A(x_i) \supset A(t)$ if t is free for x_i in A

$(\forall x_i)(A \supset B) \supset (A \supset (\forall x_i) B)$ if A contains no free occurrence of x_i

The rules of inference are:

If A and $A \supset B$, then B — modus ponens

If A then $(\forall x_i) A$ — universal generalization

The connective $\exists$ is usually added.

Resolution (Propositional)

Resolution applies to formulas in Conjunctive Normal Form (CNF), i.e.,

disjunction

$$F = (L_{1,1} \vee \ldots \vee L_{1,n_1}) \wedge \ldots \wedge (L_{k,1} \vee \ldots \vee L_{k,n_k})$$

(This formula has k clauses, each of which is a disjunction of $(n_1, \ldots n_k)$ literals).

A literal is either a propositional variable or a negated propositional variable (e.g., P_1; $\sim P_2$) $(A_1; \neg A_2)$

CNF formulas are often represented in set notation, i.e., a CNF formula is a <u>set</u> of clauses, each of which is a <u>set</u> of literals.

<u>Definition</u>. Resolvent