2013-02-07

Note Title 2013-02-07

Theorem 9.5. If $A \in F(P_0)$ and A is a tautology, then $\vdash_{P_0} A$.

(The completeness [Post-completeness] theorem of the propositional calculus.)

First, <u>Lemma 9.2</u>. Let $A \in F(P_0)$ and let $p_1, \dots, p_k$ be the propositional variables occuring in A. Consider each row of the truth table for A and for each p_i write the formula B_i as follows. If in that row p_i has t assigned to it, then B_i is just p_i, but if p_i has f assigned to it, then B_i is $\sim p_i$. Define a formula A' in a similar way; if A is assigned t, then A' is just A; if A is assigned f, then A' is $\sim A$. Then,

$$B_1, \ldots, B_k \vdash_{p_0} A'.$$

Before the proof of the lemma, an example:

$A \; (\in F(P_0)) = (p_1 \supset (p_2 \supset p_1))$

1 $\Rightarrow$

rule 2 $\Rightarrow$

T.T.:

p_1	p_2	$(p_2 \supset p_1)$	$(p_1 \supset (p_2 \supset p_1))$
t	t	t	t
t	f	t	t
f	t	f	t
f	f	t	t

Let us build the formulas B_1, B_2, and A' for row 1 of the tt:

$B_1 = p_1$, $B_2 = p_2$, $A' = A$, so the lemma claims that

$$B_1, B_2 \vdash A$$

$$p_1, p_2 \vdash (p_1 \supset (p_2 \supset p_1))$$

Row 2: $p_1, \sim p_2 \vdash (p_1 \supset (p_2 \supset p_1))$

Row 3: $\sim p_1, p_2 \vdash (p_1 \supset (p_2 \supset p_1))$

Row 4: $\sim p_1, \sim p_2 \vdash (p_1 \supset (p_2 \supset p_1))$

Example 2: $A: (p_2 \supset p_1)$, TT:

p_1	p_2	$(p_2 \supset p_1)$
t	t	t
t	f	t
f	t	f
f	f	t

row 1: $p_1, p_2 \vdash (p_2 \supset p_1)$

row 2: $p_1, \sim p_2 \vdash (p_2 \supset p_1)$

row 3: $\sim p_1, p_2 \vdash \sim (p_2 \supset p_1)$

row 4: $\sim p_1, \sim p_2 \vdash (p_2 \supset p_1)$

Exercise 9.16 (completion of lemma 2)

(b) suppose A is assigned t and B is assigned t

and C is assigned t Then,

(1) $B_1, \ldots, B_k \vdash B$ by ind. assumption

(2) $B_1, \ldots, B_k \vdash C$ " " "

(3) $B_1, \ldots, B_k \vdash (C \supset (B \supset C))$ axiom 1

(4) $B_1, \ldots, B_k \vdash (B \supset C)$ mp on (2) and (3)

(c) A is assigned t, B is assigned f, and C is assigned t. Then

(1) $B_1, \ldots B_r \vdash \sim B$ by ind ass.

(2) $B_1, \ldots, B_r \vdash C$ by ind. ass.

(3) $B_1, \ldots, B_r \vdash (C \supset (B \supset C))$ axiom 1

(4) $B_1, \ldots, B_r \vdash (B \supset C)$ mp on (2) & (3)

[Cases (b) and (c) "reflect" the fact that an implication is true, when its conclusion is true; the value of the premise does not matter.)

(d) A is assigned t, B is assigned f, and C is assigned f

$$\left.\begin{array}{ll} (1) & B_1, \ldots, B_k \vdash \sim B \\ (2) & B_1, \ldots, B_k \vdash \sim C \end{array}\right\} \text{ ind. ass.}$$

(3) $B_1, \ldots, B_k \vdash (\sim B \supset (B \supset C))$ Lemma 9.1 (3)

(4) $B_1, \ldots, B_k \vdash (B \supset C)$ mp on (1), (3)

$\square$

Proof of Theorem 9.5 (If $A \in F(P_0)$ and A is a tautology, then $\vdash_{P_0} A$)

Let A be a tautology

Let $p_1, \ldots, p_k$ be the propositional variables in A.

Then, the TT for A has 2^k rows

In half of them, p_k is assigned t,

so B_k is p_k, and in the other half,

p_k is assigned f, so B_k is $\sim p_k$.

So, by Lemma 9.2, in the first half of the rows,

$B_1, \ldots, B_{k-1}, p_k \vdash A$.

In the second half,

$B_1, \ldots, B_{k-1}, \sim p_k \vdash A$

Use the deduction theorem for both

cases:

$B_1, \ldots, B_{k-1} \vdash (p_k \supset A)$

$B_1, \ldots, B_{k-1} \vdash (\sim p_k \supset A)$ Lemma 9.1(8)

$B_1, \ldots, B_{k-1} \vdash ((p_k \supset A) \supset ((\sim p_k \supset A) \supset A))$

$B_1, \ldots, B_{k-1} \vdash A$ m.p. twice

Redo this k-1 more times and conclude

$\vdash A$ $\square$

Exercise 9.19
(c) due on Thursday 2/14.