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Exercise 9.9. (c)

Are the axioms of P_0 tautologies?

[Defn: If a formula C of $F(P_1)$ has the tv (truth value) t

(true) assigned to it for all tv assignments to its

propositional variables, then we say that C is a

tautology.]

$(A \supset (B \supset A))$ (Informally, b/c we treat the formulas A, B, C as truth values)

A	B	$(\sim B \vee A) \equiv (B \supset A)$	$(A \supset (B \supset C))$
t	f	t	t
t	f	t	t
f	t	f	t
f	f	t	t

Axiom 1 is a tautology

Axiom 2 and Axiom 3 can be verified similarly.

Exercise 9.8(d). Show: If A is a tautology and $(A \supset B)$ is a tautology, then B is also a tautology

If $(A \supset B)$ is a tautology, then for all tv assignments, $(A \supset B)$ has tv t. In particular, it has tv t for the assignments for which

A has tv t. If A has tv t, then in order for $(A \supset B)$ to have tv t, then B has tv t, because of the truth table for implication. So, since A is a tautology, then B is also a tautology.

p_0	$p_0 \supset p_1$	p_1
t	t	t
b/c tautology	b/c tautology	b/c of the defn. of implication, " " the truth table for implication

Theorem 9.2 [The soundness of P_0; or: "the soundness of the propositional calculus"].

If $\vdash_{P_0} A$, then A is a tautology.

Proof by complete induction on the length of the derivation (proof) of A. [proof $\equiv$ derivation]

Basis. (derivation of length 1). Let D be a theorem of P_0 with a proof of length 1. Then, by definition of proof [from the axioms and rules of inference], D is an axiom. By exercise 9.9(c), the axioms are tautologies, and the basis case is proved.

Inductive step. Let B be a theorem with a proof of length $k > 1$. Then, two cases. (1). B is an axiom. Then, just as before, B is a tautology.

(2) If B is not an axiom, it follows from two previous formulas in the derivation by means of the rule of inference modus ponens (which is the only rule of inference of P_0). So, the two previous formulas have the form A and $(A \supset B)$. By the ind. hypothesis (since A and $(A \supset B)$ are derived formulas of length $< k$), A and $(A \supset B)$ are

tautologies. By Exercise 9.9(d), B is also a tautology.

Exercise 9.14 (Theory $\mathcal{Y}$)

$L(\mathcal{Y}) = L(P_0)$ (the prop. vars, parentheses $\supset$, $\sim$, ..)

$((A \supset B) \supset (A \supset A))$ the axiom (scheme)

$\{A, (A \supset B)\} \to B$ the rule of inference (m.p.)

Part (e). Is every theorem in $\mathcal{Y}$ a tautology?

Equivalently: Is the axiom of $\mathcal{Y}$ a tautology?

A	B	$A \supset B$	$A \supset A$	$((A \supset B) \supset (A \supset A))$
f	f	t	t	t
f	t	t	t	t
t	f	f	t	t
t	t	t	t	t

Yes!

Part (b)

If $\vdash_Y C$, then C is of the form

$$((A \supset B) \supset (A \supset A)) \quad \text{or} \quad ((A \supset B) \supset (A \supset B)).$$

1. $\vdash_Y ((A \supset B) \supset (A \supset A))$ axiom
2. $\vdash_Y (((A \supset B) \supset (A \supset A)) \supset ((A \supset B) \supset (A \supset B)))$ axiom
3. $\vdash_Y ((A \supset B) \supset (A \supset B))$ mpon 1, 2

To conclude, argue by cases — you will show that no other formulas can be derived.

Part (c)

$\vdash_{\mathcal{Y}} (A \supset A)$?

No, b/c it does not have the form of derivable formulas given in part (b)

Part (d)

If $C \in F(\mathcal{Y})$ and C is a tautology, is C a theorem of $\mathcal{Y}$? No – ex. $\nvdash_{\mathcal{Y}} (A \supset A)$, and $(A \supset A)$ is a tautology.

Part (e) Are P_0 and Y equal?

Recall: a theory is identified with the set of its theorems. By this defn, P_0 and Y are not equal, b/c P_0 includes $(A \supset A)$ and Y does not

Part (f) Does the deduction theorem hold in Y?

No, b/c $A \vdash_Y A$ (the defn of proof from hypotheses & axioms is unchanged), but

$\vdash_Y (A \supset A)$ is false.

Part (g) Is Y consistent?

Yes, b/c negating the only theorems of Y (from part (c)) gets formulas that are not (structurally) in the form of a theorem of Y.

$\sim((A \supset B) \supset (A \supset A))$ is not an instance of $((A \supset B) \supset (A \supset A))$ or of $((A \supset B) \supset (A \supset B))$, and $\sim((A \supset B) \supset (A \supset B))$ is not an instance of $((A \supset B) \supset A \supset A))$ or of $((A \supset B) \supset (A \supset B))$.

Part (h): Does Y have a solvable decision problem? I.e., Could you write a program that,

when given a formula of $L(Y)$, say C,
decides whether C is a theorem of Y?