Proofs

- A **proof** is a mechanically derivable demonstration that a formula logically follows from a knowledge base.
- Given a proof procedure, $KB \vdash g$ means $g$ can be derived from knowledge base $KB$.
- Recall $KB \models g$ means $g$ is true in all models of $KB$.
- A proof procedure is **sound** if $KB \vdash g$ implies $KB \models g$.
- A proof procedure is **complete** if $KB \models g$ implies $KB \vdash g$. 
One rule of derivation, a generalized form of *modus ponens*:

If “$h \leftarrow b_1 \land \ldots \land b_m$” is a clause in the knowledge base, and each $b_i$ has been derived, then $h$ can be derived.

This is forward chaining on this clause.

(This rule also covers the case when $m = 0$.)
Bottom-up proof procedure

KB ⊢ g if g ∈ C at the end of this procedure:

\[ C := \{\}; \]
\[ \text{repeat} \]
\[ \quad \textbf{select} \ \text{clause} \ "h \leftarrow b_1 \land \ldots \land b_m" \ \text{in} \ KB \ \text{such that} \]
\[ b_i \in C \ \text{for all} \ i, \ \text{and} \]
\[ h \notin C; \]
\[ C := C \cup \{h\} \]
\[ \textbf{until} \ \text{no more clauses can be selected}. \]
Example

\begin{align*}
a & \leftarrow b \land c. \\
a & \leftarrow e \land f. \\
b & \leftarrow f \land k. \\
c & \leftarrow e. \\
d & \leftarrow k. \\
e & . \\
f & \leftarrow j \land e. \\
f & \leftarrow c. \\
j & \leftarrow c. \\
\end{align*}
Soundness of bottom-up proof procedure

If \( KB \vdash g \) then \( KB \models g \).

- Suppose there is a \( g \) such that \( KB \vdash g \) and \( KB \models \not\models g \).
- Then there must be a first atom added to \( C \) that isn’t true in every model of \( KB \). Call it \( h \). Suppose \( h \) isn’t true in model \( I \) of \( KB \).
- There must be a clause in \( KB \) of form

\[
h \leftarrow b_1 \land \ldots \land b_m
\]

Each \( b_i \) is true in \( I \). \( h \) is false in \( I \). So this clause is false in \( I \). Therefore \( I \) isn’t a model of \( KB \).
- Contradiction.
The $C$ generated at the end of the bottom-up algorithm is called a **fixed point**.

Let $I$ be the interpretation in which every element of the fixed point is true and every other atom is false.

$I$ is a model of $KB$.
Proof: suppose $h \leftarrow b_1 \land \ldots \land b_m$ in $KB$ is false in $I$. Then $h$ is false and each $b_i$ is true in $I$. Thus $h$ can be added to $C$. Contradiction to $C$ being the fixed point.

$I$ is called a **Minimal Model**.
If $KB \models g$ then $KB \vdash g$.

- Suppose $KB \models g$. Then $g$ is true in all models of $KB$.
- Thus $g$ is true in the minimal model.
- Thus $g$ is in the fixed point.
- Thus $g$ is generated by the bottom up algorithm.
- Thus $KB \vdash g$. 