Predicate Logic (or: First-order Logic [FOL])

or First-order predicate logic

or Predicate Calculus

$P_k$ is the arity of a predicate (symbol)

arity is the number of arguments
Terms
Ex. 43 matrix: \((Q(x) \cup (P(f(x), z) \land \theta(a)) \land R(x, z, q(x)))\)

Shortcut:
\[I_{a}(x) = x^{a}\]
\[I_{a}(f) = f^{a}\]
\[I_{a}(P) = P^{a}\]
\[I_{a}(b) = b^{a}\]
\[ P(x, f(x)) \]

\[ \mathcal{I} / P(x, f(x)) = \text{student (coupon code)} \]

\[ = P^a (x^a, f^a (x^a)) \]

\[ = \prec (2, \text{succ (2)}) = \prec (2, 3) \]

\[ = 2 < 3 = 0 \text{ (f)} \]
If $F$ has the form $F = \forall x \in A$, then

$$\sigma(F) = \begin{cases} 1, & \text{if for all } u \in \mathcal{U}_A, \mathcal{A}[x/u](g) \neq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $F$ has the form $F = \exists x \in A$, then

$$\sigma(F) = \begin{cases} 1, & \text{if for some } u \in \mathcal{U}_A, \mathcal{A}[x/u](g) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$
\( A \models F \) if \( \varphi(F) = 1 \)

If \( \varphi(F) = 1 \) for every suitable structure \( \mathcal{A} \), then \( F \) is valid, written \( \vdash F \).

If there is some suitable structure \( \mathcal{A} \) for which \( \varphi(F) = 1 \), then \( F \) is satisfiable.

If there is no model for \( F \) (i.e., no suitable structure \( \mathcal{A} \) s.t. \( \varphi(F) = 1 \)), then \( F \) is unsatisfiable.
The $F$ is unsatisfiable (or a contradiction).

Exercise 4.4

$F: \forall x \exists y \, P(x, y, f(z))$

A model for $F$ is $\mathfrak{A}(U_a, I_a)$, where

$U_a = \{ x \}$

$P^a \subseteq \{ (x, c, c) \}$

$z^a = c$

$f^a = c \circ c$
On interpretation $B$ s.t. $B(f) = 0$

$V^B = \{ c \}$

$z = c$

$f^B = c \rightarrow c$

$p^B = \frac{1}{3}$

Exercise 4.5

$U = \{ 1, 2, 3 \}$

$a$ is a model of $F_1$, $F_2$, and $F_3$

$F_1 + F_2 + F_3$

$P = \frac{1}{3} \{ (1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 1) \}$
$F_2 + F_3$ $B$ is a model of $F_2, F_3$, not $F_1$

$P = \{ 1 \}$

$F_1, F_3$ $C$ is a model of $F_1, F_3$, not $F_2$

$P = \{ (1, 1), (2, 2), (3, 3), (1, 2) \}$

$F_1, F_2$ $D$ is a model of $F_1, F_2$, not $F_3$

$\{ (1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2) \}$
Exercise 4.6

Syntax is changed by adding

if \( t_1 \) and \( t_2 \) are terms, then \( t_1 = t_2 \) is a

formula

Semantics are changed by adding:

if \( F \) has the form \( t_1 = t_2 \), then

\[
\alpha(F) = \begin{cases} 
1 & \text{if } \alpha(t_1) = \alpha(t_2) \\
0 & \text{otherwise}
\end{cases}
\]
Exercise 47

For part (a), the question is whether the system of inequalities
\[ x < y \\
2 < y \] (Yes)
\[ x < 2 \\
x = 1 \\
z = 2x \quad x = 1 \]
\[ y = 3 \]

(b)
\[ y = x + 1 \\
y = z + 1 \\
z = x + 1 \\
x \neq z + 1 \] (No)
Exercise 44

\[ F = \forall x \in \mathcal{E} \left( x, x \right) \land \exists x \exists y \exists z \]

\[ \left( \neg \in \left( x, y \right) \land \neg \in \left( x, z \right) \land \in \left( y, z \right) \right) \]

Here is instead a formula whose models have universes of cardinality exactly 3:

(Note: \( t_1 \neq t_2 \) is an abbreviation for \( \neg \left( t_1 = t_2 \right) \))

\[ \exists x \exists y \exists z \forall u \left( x \neq y \land y \neq z \land x \neq z \land x = u \lor y = u \lor z = u \right) \]

\( a \geq 3 \) individuals

at most three individuals
(individual means element of the universe)

Exercise 57

\[ \forall x \forall y \left( (x = y) \land (y \neq 2) \lor (x = 2) \right) \]

Exercise 52

(a) \[ \forall x \forall y \ (P(x, y) \lor P(y, x)) \]

(b) \[ \forall x \forall y \ ((f(x) = f(y)) \Rightarrow (x = y)) \] \text{ if } \text{ is an injective function}

(c) \[ \forall y \exists x \ (f(x) = y) \] \text{ if } \text{ is onto or surjective}
A function both injective and surjective is called a bijection.

Exercise 53

\[ F = \forall x \forall y \forall z \left[ f(x, f(y, z)) = f(f(x, y), z) \right] \]

\[ \land \exists x \left[ \forall y \left( f(x, y) = y \right) \right] \]  (neutral element)

\[ \land \forall y \, \exists z \left( f(y, z) = x \right) \]  (inverse)
Exercise 54.

\[ F = \text{Is Empty} (\text{null stack}) \]
\[ \land \forall x \forall y (\neg \text{Is Empty} (\text{push} (x, y))) \]
\[ \land \forall x \forall y (\top \implies \text{top} (\text{push} (x, y)) = x) \]
\[ \land \forall x \forall y (\top \implies \text{top} (\text{push} (x, y)) = y) \]
\[ \land \forall x (\neg \text{Is Empty} (x) \implies \text{push} (\text{top} (x), \text{pop} (x)) = x) \) \]