Lemma 9.1 (1) - Alternate proof

1. \((A \land B), (B \land C), A \rightarrow A\) hypothesis
2. \(\therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad 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6. \((A > B), (B > C)\)  \vdash  (A > C)\)  Deduction Theorem

7. \((A > B)\)

8. \((A > B) \vdash (B > C) \supset (A > C)\)

Proof of Lemma 9.1 (3): \(\vdash (\neg B > (B > C))\)

1. \(\neg B \vdash \neg B\)  hypothesis

2. \(\neg B \vdash (\neg B \supset (\neg C \supset \neg B))\)  axiom 1

3. \(\neg B \vdash ((\neg C \supset \neg B) \supset (B > C))\)  axiom 3

4. \(\neg B \vdash (\neg C \supset \neg B)\)  m.p. on 1, 2
5. \( \neg B \vdash (B \rightarrow C) \) m.p. on 4, 3

6. \( \vdash (\neg B \rightarrow (B \rightarrow C)) \) Deduction Theorem

The truth table for implication:

<table>
<thead>
<tr>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( (P_1 \rightarrow P_2) )</th>
<th>( \neg (P_1 \rightarrow P_2) )</th>
<th>( \neg P_1 )</th>
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</table>
\[(A \supset (B \supset A)) \quad | \quad A \supset B \supset \neg A)\]

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>(B ⊃ A)</th>
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This shows that axiom 1 is a tautology.

Axiom 2 and axiom 3 are tautologies also.

If A is a tautology and \((A \supset B)\) is a tautology, then B is a tautology.
If $A$ is a tautology and $(\neg A \lor B)$
is a tautology, then $B$ is a tautology.

Theorem 9.2 If $\vdash A$, then $A$ is a tautology.

Proof By complete induction on the length of
the derivation of $A$.

Basis. (Length 1). Let $D$ be a theorem (of $\mathcal{L}$)
with a proof of length 1, so $D$ is an axiom.

By exercise 9.2 (1), $D$ is a tautology.
Inductive Step. Let $B$ be a theorem with a proof of length $k > 1$. If $B$ is an axiom, then the argument of the basis case still holds. If $B$ is not an axiom, then $B$ follows from previous formulas in the derivation using modus ponens. The previous formulas have the form $A$ and $(A \rightarrow B)$. By exercise 9.9/1(d), $B$ is a tautology. \[ \square \]
Exercise 9.14

\[ L(Y) = L(P_b) \quad \text{language} \]

\[ (A \supset B) \supset (A \supset A) \quad \text{axiom} \]

\[ \{ A, (A \supset B) \} \rightarrow B \quad \text{rule of inference (mp)} \]

(a) The axiom is a tautology as one can check by \( T \)!

\[
\begin{array}{cccccc}
A & B & A \supset B & A \supset A & (A \supset B) \supset (A \supset A) \\
T & T & T & T & T \\
T & F & T & T & T \\
F & T & T & T & T \\
F & F & T & T & T \\
F & T & T & T & T \\
T & T & T & T & T \\
T & T & T & T & T \\
\end{array}
\]
m p preserves total order as shown in Exercise 9.9 (d).

So, yes

1. \( (A \cap B) > (A \cup A) \) \hspace{1cm} \text{axiom}

2. \( (A \cap B) > (A \cup A) \) \rightarrow \( (A \cap B) > (A \cap B) \) \hspace{1cm} \text{axiom with}\n
3. \( (A \cap B) > (A \cap B) \) \hspace{1cm} \text{mp on 1, 2}
Case 1. Let \( \overline{A} \) be an axiom and \( (\overline{A} \rightarrow \overline{B}) \) be also an axiom.

\[
\begin{align*}
\{ \overline{A} , (\overline{A} \rightarrow \overline{B}) \} \rightarrow & \quad \overline{B} \\
(\overline{A} \rightarrow \overline{B}) , (\overline{A} \rightarrow \overline{A}) \rightarrow & \quad (\overline{A} \rightarrow \overline{B}) \rightarrow (\overline{A} \rightarrow \overline{B})
\end{align*}
\]
Exercises 9.6.

The converse of the Deduction Theorem is:

If $B_1, \ldots, B_{k-1} \vdash_{p_0} (B_k \rightarrow c)$ then $B_1, \ldots, B_{k-1}, B_k \vdash c$.

Proof:

1. $B_1, \ldots, B_{k-1} \vdash_{p_0} (B_k \rightarrow c)$ given

2. $B_1, \ldots, B_{k-1}, B_k \vdash_{p_0} (B_k \rightarrow c)$ def. of derivation

reliance for hypothesis
Comment: This is a formal way of describing what people mean by
"the propositional calculus is monotonic?"

3 $B_1, \ldots, B_{k-1}, B_k \vdash \top, B_k$ hypothesis
4 $B_1, \ldots, B_{k-1}, B_k \vdash C$ m.p. on 2, 3
Recall Theorem 9.2: If $\vdash_{\text{Po}} A$, then $A$ is a tautology. (The soundness of the propositional calculus; the propositional calculus is sound.)

Theorem 9.5: If $A \in \mathcal{F}(\text{Po})$ and $A$ is a tautology, then $\vdash_{\text{Po}} A$.

(The propositional calculus is complete.)
Lemma 9.2 Let $A \in F(p_o)$ and let $p_1, \ldots, p_r$ be the propositional variables that occur in $A$. Consider each row of the table for $A$ and for each $p_i$ write $B_i$ as follows: if $p_i$ is $1$, then $B_i = p_i$; otherwise, $B_i = \neg p_i$. Similarly, let $A'\equiv A$ if $A$ is $1$ in that row of the table and let $A'\equiv \neg A$ otherwise. Then, $B_1, \ldots, B_r \vdash p_o A'$. 
Example of the construction:

\[ A = (p_1 \rightarrow (p_2 \Rightarrow p_1)) \]

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Example: \( A = (P_2 > P_1) \)

\[
\begin{array}{c}
\begin{array}{c}
l_1, P_2 \vdash (P_2 > P_1) \\
l_1, \neg P_2 + (P_2 > P_1) \\
\neg l_1, P_2 \vdash \neg (P_2 > P_1) \\
\neg l_1, \neg P_2 + (P_2 > P_1)
\end{array}

\begin{array}{ccc|c}
p_1 & p_2 & P_2 > P_1 \\
1 & 1 & \text{t} \\
1 & 0 & \text{t} \\
0 & 1 & \text{f} \\
0 & 0 & \text{t}
\end{array}
\end{array}
\]
Proof (by induction on the number of connectives) (Note: Let \( k \) be the number of propositional variables in \( A \))

**Base** \( \{n = 0\} \)
The formula has the form \( p_i \ (= A) \)

On \( k \):

\[ p_i \lor p_i \]
\[ \neg p_i \lor \neg p_i \]
Inductive case. At least one proposition connecting. Consider two cases:

A has the form \( \neg C \) — see below.

A has the form \((B \supset C)\) — see below for deriving — (a)

(b) so pose that \( A \) is assigned \( \top \)

\( B \) is assigned \( \bot \), and \( C \) is assigned \( \bot \)

By inductive assumption:

(1) \( B_1, \ldots, B_k \vdash B \)

(2) \( B_1, \ldots, B_k \vdash C \)
(3) \( B, \ldots, B_k \vdash \lnot \lnot (B > c) \)  \( \text{axiom 1} \)

(4) \( B, \ldots, B_n \vdash (B > c) \)  \( \text{on } p \text{ on } 2, 3 \)

(5) \( A \) is assigned \( \top \), \( B \) is assigned \( \top \), and \( C \) is assigned \( \top \).  \( \text{Then:} \)

(1) \( B, \ldots, B_k \vdash a \vdash B \)  \( \text{ind. assumption} \)

(2) \( B, \ldots, B_k \vdash \top \)

(3) \( B, \ldots, B_k \vdash (\top (B > c)) \)  \( \text{axiom 1} \)

(4) \( B, \ldots, B_n \vdash (B > c) \)  \( \text{on } p \text{ on } 2, 3 \)
(a) $A$ is $t$, $B$ is $f$, $C$ is $f$.

(1) $B_1 \ldots B_k + \alpha B \\
\text{Ind. assumption}$

(2) $B_1 \ldots B_k + \alpha C$

(3) $B_1 \ldots B_k + (\alpha B \geq (B > C))$ \hspace{1cm} \text{Lemma 9.1 (3)}

(4) $B_1 \ldots B_k + (B > C)$ \hspace{1cm} \text{mp on (1) and (2)}

Show with Lemma 9.2
Proof of Theorem 9.5: If $A \in P_0$ and $A$ is a tautology, then $\vdash P_0 A$.

If $A$ is a tautology, then it is assigned $T$ in every row of its truth table.

Let $p_1, \ldots, p_k$ be the propositions of $A$.

The truth table of $A$ has $2^k$ rows.

In half of them, $p_k$ is assigned $T$, so
$B_k$ (of Lemma 9.2) is $P_k$, and

$M_B, \ldots, B_{k-1}, P_k \vdash A$ (by Lemma 9.2)

In the other half of the rows of $M_B, P_k$ is!

$M_B, \ldots, B_{k-1}, A \vdash A$ (by Lemma 9.2)

$M_B, \ldots, B_{k-1}, P_k \vdash (P_k \Rightarrow A)$ and then on (1)

(2) $B_1, \ldots, B_{k-1} \vdash (\neg P_k \Rightarrow A)$ and then on (2)
(5) \[ B_1 \ldots \ldots, B_{k-1} \vdash ((\neg \mu \triangleright A) \circ (\neg \nu \triangleright \neg A) \triangleright A) \]

Lemma 9.1 (8)

Use m, p, twice (5, 3, 4):

(7) \[ B_1 \ldots \ldots, B_{k-1} \vdash A \]

Do this (1-6) \( k-1 \) more times, and obtain,

\[ \vdash A \]