Yes where Ch. 9 (Propositional Calculus)

Lemma 9.1 (1) - alternate proof

1. \((A \lor B), (B \lor C), A \lor A\)  
   hypothesis

2. \((A \lor B)\)  

3. \((A \lor A)\)  
   m.p. on 1 and 2

4. \((B \lor C)\)  
   m.p. on 3 and 4

5. \((C)\)
6. \((A > B), (B > C) \vdash (A > C)\) \textit{Deduction Theorem}

7. \((A > B) \vdash (B > (A > C))\) \textit{Deduction Theorem}

8. \((A > B) \vdash (B > (A > C))\) \textit{Deduction Theorem}

\textbf{Proof of Lemma 9.1 (3)}: \(\vdash (\neg B > (B > C))\)

1. \(\neg B \vdash \neg B\) \textit{hypothesis}

2. \(\neg B \vdash (\neg B > (\neg C \neg B))\) \textit{axiom 1}

3. \(\neg B \vdash ((\neg C > \neg B) > (B > C))\) \textit{axiom 3}

4. \(\neg B \vdash (\neg C > \neg B)\) \textit{m.p. on 1, 2}
5. \( \neg B \vdash (B \Rightarrow C) \) m.p. on 4, 3
6. \( \vdash (\neg B \Rightarrow (B \Rightarrow C)) \) Deduction Theorem

The truth table for implication:

<table>
<thead>
<tr>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( (P_1 \Rightarrow P_2) )</th>
<th>( (\neg P_1 \lor \neg P_2) )</th>
<th>( \neg P_1 )</th>
</tr>
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\( \neg P_1 \):
\[(A \rightarrow (B > A)) \quad \begin{array}{c|c|c|c}
A & B & (B > A) \\
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\text{t} & \text{t} & \text{t} \\
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\end{array}\]

This show that axion 1 is a tautology.

Axion 2 and axion 3 are tautologies also.

If \(A\) is a tautology and \((A \rightarrow B)\) is a tautology, then \(B\) is a tautology.
If $A$ is a tautology and $(\neg A \lor B)$ is a tautology, then $B$ is a tautology.

**Theorem 9.2** If $\vdash A$, then $A$ is a tautology.

**Proof.** By complete induction on the length of the derivation of $A$.

**Basis.** (Length 1). Let $D$ be a theorem (of $\mathcal{L}$) with a proof of length 1. So $D$ is an axiom.

By exercise 9.9. (c), $D$ is a tautology.
Inductive step. Let $B$ be a theorem with a proof of length $k > 1$. If $B$ is an axiom, then the argument of the basis case still holds.

If $B$ is not an axiom, then $B$ follows from previous formulas in the derivation using modus ponens. The previous formulas have the form $A$ and $(A \Rightarrow B)$. By exercise 9.9 (d), $B$ is a tautology.
Exercise 9.4

\[ L(Y) = L(P_0) \] language

\[ (A \supset B) \supset (A \supset A) \] axiom

\[ \{ A, (A \supset B) \} \rightarrow B \] rule of inference (in \( p \))

(a) The axiom is a tautology as one can check by \( TT! \):

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<tr>
<th></th>
<th>A</th>
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<th>(A \supset B) \supset (A \supset A)</th>
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mp preserves implication as shown in Exercise 1.9 (d).

So: yes

(a) 1. \( \vdash ((A \rightarrow B) \rightarrow (A \rightarrow A)) \)  
   \[ \text{axiom} \]

2. \( \vdash ((A \rightarrow B) \rightarrow (A \rightarrow A)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow B)) \)  
   \[ \text{axiom} \]

3. \( \vdash ((A \rightarrow B) \rightarrow (A \rightarrow B)) \)  
   \[ \text{mp on 1, 2} \]

\( A \rightarrow A \lor B \rightarrow A \lor C \rightarrow A \lor C \)
Case 1: Let \( \overline{A} \) be an axiom and \((\overline{A} \supset \overline{B})\) be also an axiom.

\[
\{ \overline{A}, (\overline{A} \supset \overline{B}) \} \vdash \overline{B} \quad (\text{by axiom})
\]

\[
(\overline{A} \supset \overline{B}, (\overline{A} \supset \overline{B})) \vdash (\overline{A} \supset \overline{B}) \vdash (A \supset B)
\]