Steffen L. Lauritzen. Graphical Models. Oxford University Press, 1996.

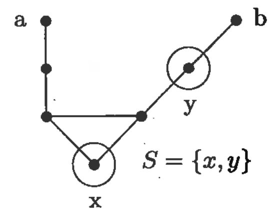

Fig. 3.2. The moral graph of the smallest ancestral set in the graph of Fig. 3.1 containing $\{a\} \cup \{b\} \cup S$. Clearly S separates a from b in this graph, implying $a \perp\!\!\!\perp b \mid S$.

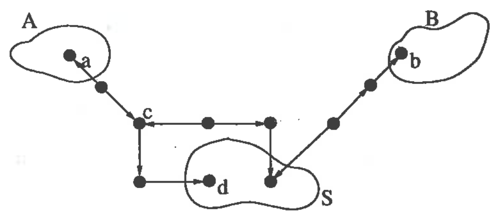

Fig. 3.3. Example of an active chain from A to B. The path from c to d is not part of the chain, but indicates that c must have descendants in S.

An alternative formulation of the directed global Markov property was given by Pearl (1986*a*, 1986*b*) with a full formal treatment in Verma and Pearl (1990*a*, 1990*b*). A chain π from a to b in a directed, acyclic graph $\mathcal{G}$ is said to be *blocked* by S, if it contains a vertex $\gamma \in \pi$ such that either

- $\gamma \in S$ and arrows of π do not meet head-to-head at γ, or
- $\gamma \notin S$ nor has γ any descendants in S, and arrows of π do meet head-to-head at γ.

A chain that is not blocked by S is said to be *active*. Two subsets A and B are now said to be *d-separated* by S if all chains from A to B are blocked by S. We then have

Proposition 3.25 *Let A, B and S be disjoint subsets of a directed, acyclic graph $\mathcal{G}$. Then S d-separates A from B if and only if S separates A from B in $(\mathcal{G}_{\mathrm{An}(A \cup B \cup S)})^m$.*

Proof: Suppose S does not d-separate A from B. Then there is an active chain from A to B such as, for example, indicated in Fig. 3.3. All vertices in this chain must lie within $\mathrm{An}(A \cup B \cup S)$. This follows because if the arrows

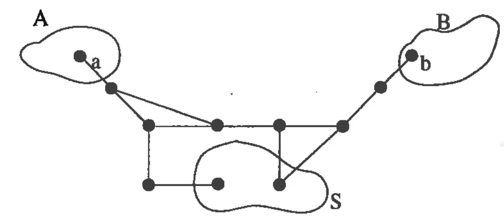

Fig. 3.4. The moral graph corresponding to the active chain in $\mathcal{G}$.

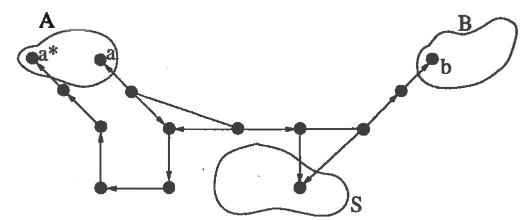

Fig. 3.5. The chain in the graph $(\mathcal{G}_{\mathrm{An}(A \cup B \cup S)})^m$ makes it possible to construct an active chain in $\mathcal{G}$ from A to B.

meet head-to-head at some vertex γ, either $\gamma \in S$ or γ has descendants in S. And if not, either of the subpaths away from γ either meets another arrow, in which case γ has descendants in S, or leads all the way to A or B. Each of these head-to-head meetings will give rise to a marriage in the moral graph such as illustrated in Fig. 3.4, thereby creating a chain from A to B in $(\mathcal{G}_{\mathrm{An}(A \cup B \cup S)})^m$, circumventing S.

Suppose conversely that A is not separated from B in $(\mathcal{G}_{\mathrm{An}(A \cup B \cup S)})^m$. Then there is a chain in this graph that circumvents S. The chain has pieces that correspond to edges in the original graph and pieces that correspond to marriages. Each marriage is a consequence of a meeting of arrows head-to-head at some vertex γ. If γ is in S or it has descendants in S, the meeting does not block the chain. If not, γ must have descendants in A or B, since the ancestral set was smallest. In the latter case, a new chain can be created with one head-to-head meeting fewer, using the line of descent, such as illustrated in Fig. 3.5. Continuing this substitution process eventually leads to an active chain from A to B and the proof is complete. $\square$