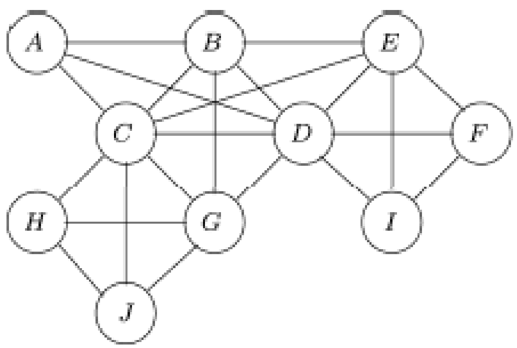
Note Title 3/20/2009

Theorem 4.4 If the undirected graph G is triangulated, then the cliques of G can be organized into a join tree

The proof consists of an algorithm to construct a join tree from G.

<u>Defn.</u> A join <u>graph</u> of $G=(V,E)$ (where G is a triangulated undirected graph) is a weighted undirected graph $J=(U,F,w)$, where

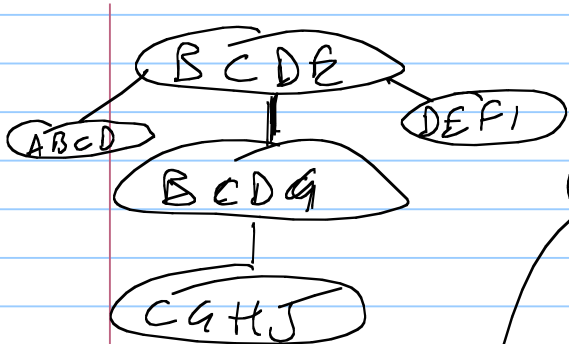
- U is the set of cliques of G $\quad (U=\{\cdots u_i \cdots\})$
- $\{u_i,u_j\}\in F$ iff $u_i \cap u_j \neq \{\}$
- $w(\{u_i,u_j\}) = |u_i \cap u_j|$

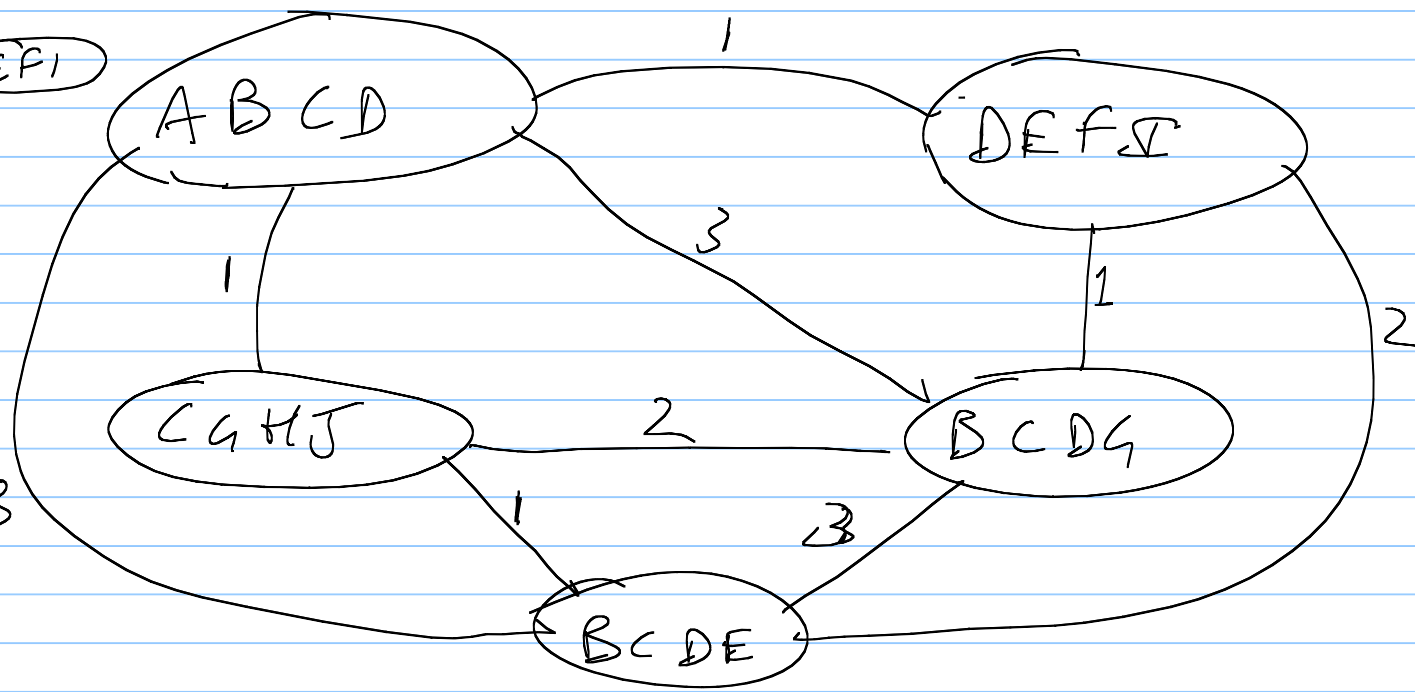


Ex. Find the cliques w/ alg. 4.1

$\langle A, F, I, H, J, G, B, C, D, E \rangle$

The junction graph for this is here





Note: in class, the letters T, G, and J were mixed up!

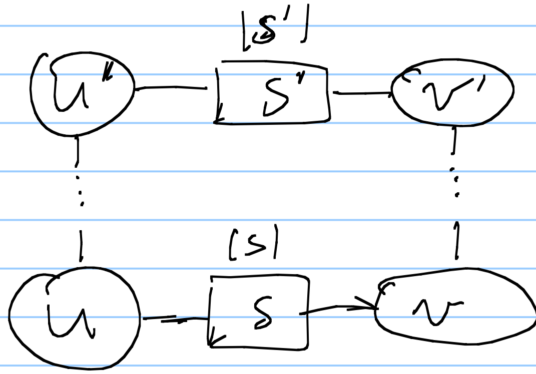
Theorem: T is a join tree of the triangulated undirected graph G iff T is a maximal spanning tree of J. [Jensen, 1988; Shibata, 1988; proof here is due to Jensen & Jensen, UAI-94].

Proof

First: proof sketch. Replace an edge (u_i, u_j) with a non-maximal one. The path between u_i and u_j in the new tree will have flow smaller than $|u_i \cap u_j|$ and therefore the join condition will not be satisfied.

Let T be a spanning tree of maximal weight.
Let it be constructed using Prim's algorithm (wlog), s.t.
$T_1 \subseteq T_2 \subseteq \dots \subseteq T_n = T$ is a sequence of partial spanning trees.

Assume that T is not a join tree. Then, at some stage m, we have that T_m can be extended to a j. tree T', but T_{m+1} cannot. Let (u, v) be the edge added to T_m to make it into T_{m+1}. Let $u \cap v = S$.



Consider the path between u and v in the tree through $u' — v'$

Since $|u \cap v| = |S|$, then $|S'| \geq |S|$. So Prim's algorithm could have chosen $\{u', v'\}$ in place of $\{u, v\}$, contrary to the claim

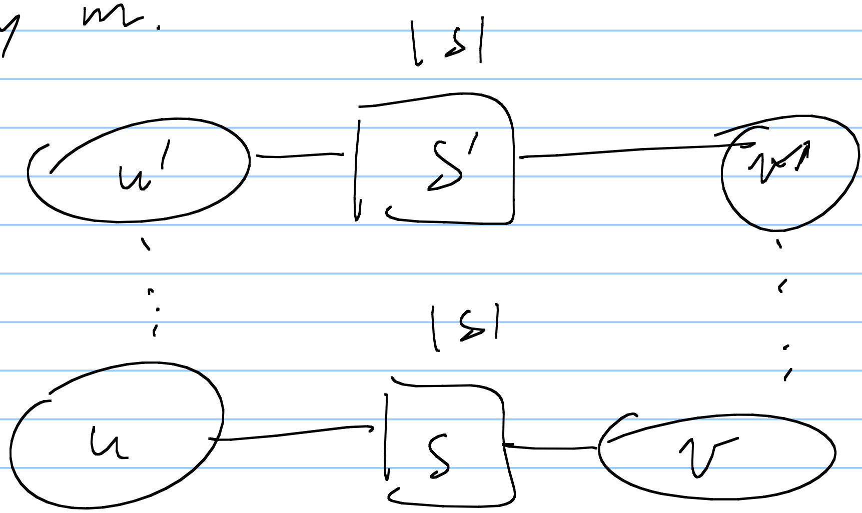
Let T be a non-maximal spanning tree.

We show that T is not a join tree.

Use the same construction, via Prim's algorithm

$T_1 \subseteq \cdots \subseteq T'$, $T' \neq T$

Let Prim try to construct T. It will fail at some stage, say m.



At that stage, Prim would not choose u'—v'. It would choose an edge like u–v with higher weight, which is absent from T. Then, the path $u \cdots u'$—$v' \cdots v$ does not the capacity to carry $u \cap v$, and so T is an impostor!