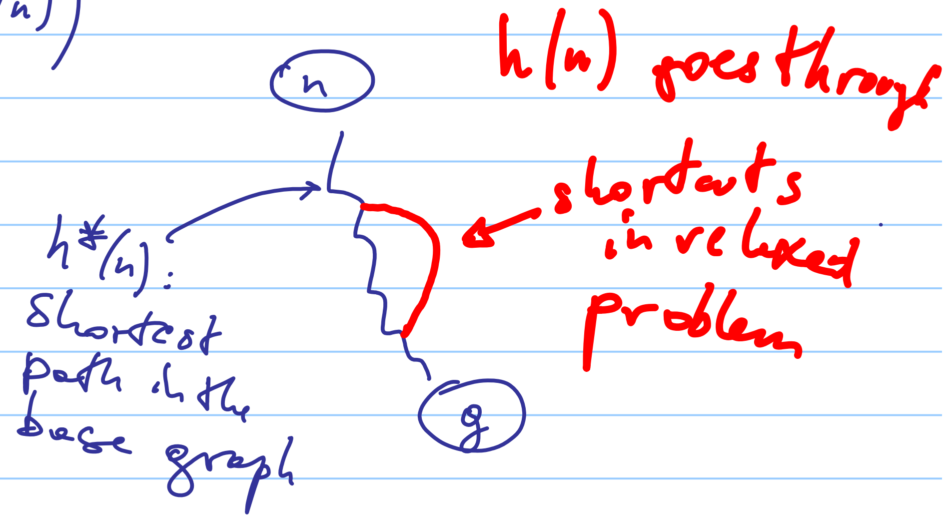
Three Properties of Heuristics Computed by Problem Relaxation

1. Heuristics computed by problem relaxation are admissible. $(h(n) \le h^*(n))$

Proof sketch:

see figure



2. Heuristics computed by problem relaxation are monotone:

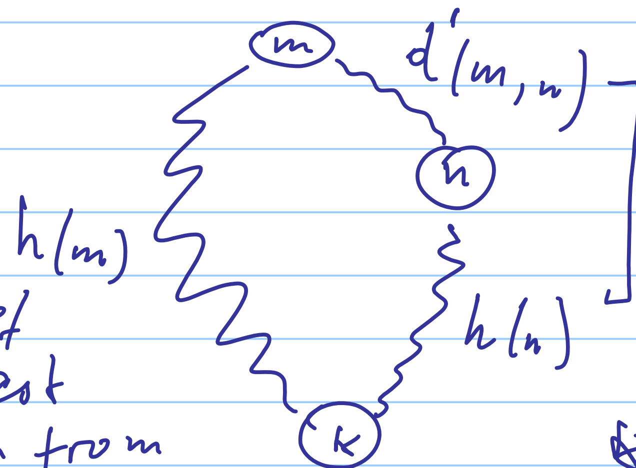
Theorem. If $h(m)$ and $h(n)$ are computed as costs of shortest path in a relaxed subproblem, then

$$h(m) - h(n) \le d(m, n).$$

Proof sketch:

See figure (showing the relaxed graph) and note that

$$\underbrace{d(m,n)}_{\text{on base problem}} \ge \underbrace{d'(m,n)}_{\text{on relaxed problem}},$$



(cost of shortest path from m to k

$d'(m,n) + h(n) =$ cost of shortest path from m to k constrained to pass through n, so

$$h(m) \le d'(m,n) + h(n) \le d(m,n) + h(n)$$

3. Consider the algorithm that solves a state space search problem in two phases:

(a) compute $h(\cdot)$ by problem relaxation using Dijkstra's algorithm (on a relaxed subproblem)

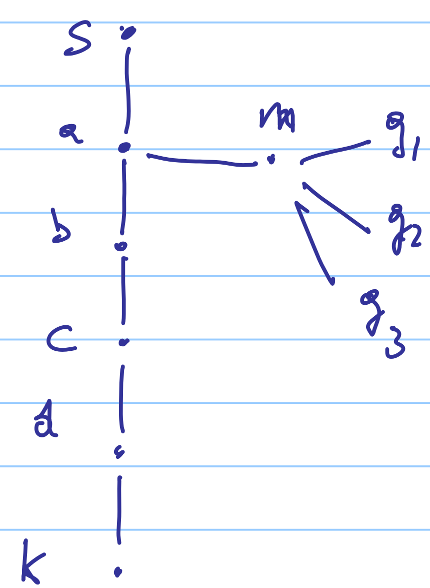
(b) use $h(\cdot)$ as A* does on the base problem.

Such algorithm (call it M) expands at least every node expanded by Dijkstra's algorithm on the same problem.

Proof.

In order to save on expanded nodes, there must be at least one node (say, m) whose heuristic $h(m)$ is large enough to prevent some nodes (say, g_i, $i = 1..k$, $k \geq 1$) from being expanded in phase (b).

<u>Example</u>



	g	h	f
s	0	0	0
a	1	0	1
b	2	0	2
c	3	0	3
m	2	4	6
g_i	3	*	*
d	4	0	4
k	5	0	5

*: not computed, b/c m is not expanded in phase (b).

The largest set of nodes of the g_i kind is characterized by the following inequality:

$$g(m) + d(m, g_i) < h^*(s) \quad (1)$$

This inequality characterizes the largest set of nodes that would be expanded by Dijkstra's algorithm (because their g value is less than $h^*(s)$), but are not expanded in phase (b) of algorithm M, because m itself is not expanded.

Inequality (1) can be rewritten as

$d(m, g_i) < h^*(s) - g(m)$ (2)

But in order for m not to be expanded in phase (b), it must be that

$g(m) + h(m) \geq h^*(s)$ (3). This can be rewritten as

$h(m) \geq h^*(s) - g(m)$ (4).

But in order to compute $h(m)$, one needs to expand, in the relevant subproblem, using Dijkstra's algorithm, <u>at least</u> all nodes closer to m than $h(m)$, i.e., all nodes h_i s.t.

$d(m, h_i) < h(m) \geq h^*(s) - g(m)$, i.e., at least the nodes h_i s.t.

$d(m, h_i) < h^*(s) - g(m)$ (5).

Comparing (5) with (2), one sees that at most the nodes for which $d(m, h_i) < h^*(s) - g(m)$ are not expanded in phase (b).

Therefore, the set of nodes that are expanded in phase (a) to compute h is a (non-necessarily strict) superset of the nodes that are not expanded in phase (b) by using the heuristic. This is what the theorem claims.

$\square$