Weighted Heuristic Anytime Search

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Basic Heuristic Searches

- Best-first Search
  Blindly follows the heuristic
- Weighted A^* Search
  For $w > 1$

$$f(n) = g(n) + w \cdot h(n)$$

Larger w yields 'greedier' searches

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Graphical Models

Definition (Graphical Model)

A tuple $\mathcal{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F}, \bigotimes \rangle$ where

1. $\boldsymbol{X} = \{X_0, \ldots, X_{n-1}\}$ is a set of *variables*
2. $\boldsymbol{D} = \{D_0, \ldots, D_{n-1}\}$ is a set of *domains*
3. $\boldsymbol{F} = \left\{f_0(\boldsymbol{X}_{S_0}), \ldots, f_{r-1}(\boldsymbol{X}_{S_{r-1}})\right\}$ is a set of *scopes*:
   - $\boldsymbol{X}_{S_i} \subseteq \boldsymbol{X}$
   - $\forall i.\ f_i : \boldsymbol{X}_{S_i} \to \mathbb{R}^+$
4. A *combination operator* $\bigotimes \in \{\sum, \prod\}$

The model $\mathcal{M}$ represents the function

$$C(X) = \bigotimes_{i=0}^{r-1} f_i(X_{S_i})$$

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Given a model $\mathcal{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F}, \bigotimes \rangle$, the most common optimization task is either *most probable explanation* or *maximum a posteriori*

MPE Find the optimal value C^*:

$$C^* = C(\boldsymbol{x}^*) = \max_{\boldsymbol{X}} \bigotimes_{i=0}^{r-1} f_i(\boldsymbol{X}_{S_i})$$

MAP Find the optimizing configuration x^*:

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Definition

The *primal graph* of a model is a graph where the vertices are the variables and edges connect variables within the same scope

Scopes

- $X_{S_0} = \{A, B\}$
- $X_{S_1} = \{A, C\}$
- $X_{S_2} = \{C, D\}$
- $X_{S_3} = \{B, D\}$
- $X_{S_4} = \{B, F\}$
- $X_{S_5} = \{E, F\}$

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AND/OR Search Graphs
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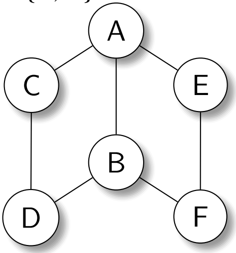

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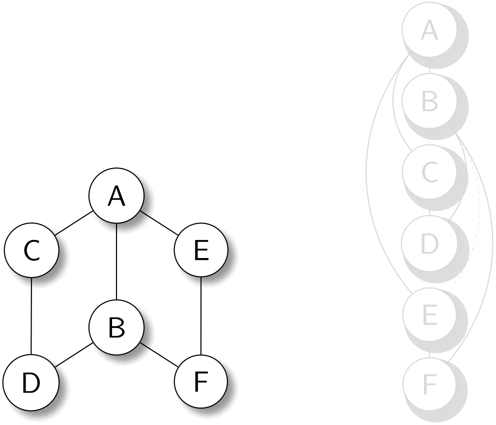
Pseudotrees

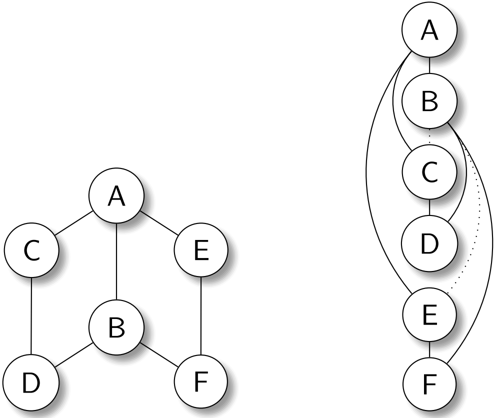


(a) Primal (b) Induced Graph

Figure: Induced graph over the natural ordering.

AND/OR Search Graphs

Pseudotrees



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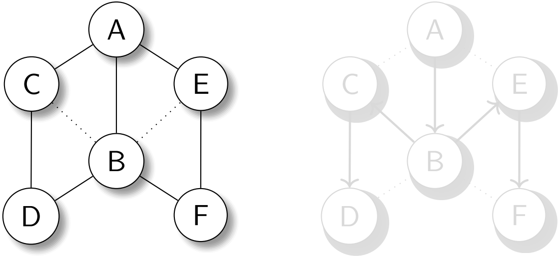
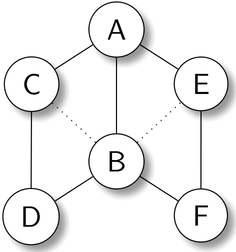


(a) Orig + Ind. Edges (b) Pseudo-tree

Figure: Pseudo-tree with edges chosen to respect the order



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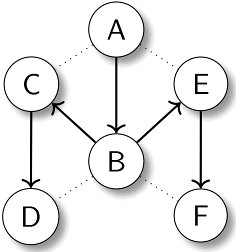

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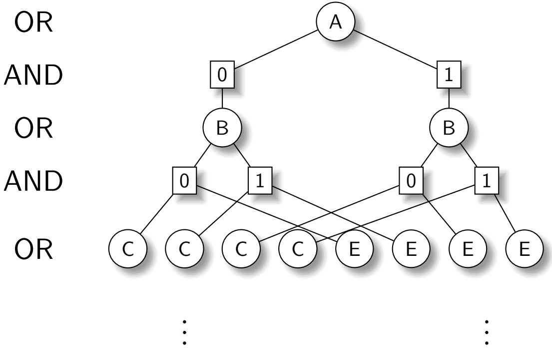

Figure: Context-Minimal AND/OR Graph For Pseudotree

Assume a graphical model $\mathcal{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F}, \bigotimes \rangle$ with primal graph G, pseudotree $\mathcal{T}$, and AND/OR search tree $\mathcal{S}_{\mathcal{T}}$

Definition

The *context-minimal AND/OR search graph*, denoted $C_{\mathcal{T}}$, is the AND/OR search graph obtained after merging all identical subproblems.

$C_{\mathcal{T}}$ is exponential in the depth of $\mathcal{T}$

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A *solution tree* T of $C_{\mathcal{T}}$ is a subtree satisfying the following conditions:

1. It contains the root of $C_{\mathcal{T}}$
2. If an internal AND node n is in T, then all the children of n are in T
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4. Every leaf in T is a terminal node

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AND/OR Best First Search

State-of-the-art A^* for AND/OR search space.

Too complicated to fit on a slide

Highlights

Input:
- Graphical Model $\mathscr{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F}, \Sigma \rangle$
- Initial weight w_0
- Pseudotree $\mathscr{T}$ rooted at X_1
- heuristic h_i (precalculated)

Output: Optimal solution to $\mathscr{M}$

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Example

Let $\mathscr{M} = \{\boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F}, \Sigma\}$ where

- $\boldsymbol{X} = \{A, B, C, D\}$
- $\boldsymbol{D} = \bigcup_{s \in \boldsymbol{X}} \{0, 1\}_s$
- $\boldsymbol{F}$ is given by the following tables:

A	B	$f(A,B)$
0	0	4
0	1	1
1	0	3
1	1	1

B	C	$f(B,C)$
0	0	3
0	1	2
1	0	2
1	1	1

B	$f(B)$
0	1
1	9

A	B	$f(A,B)$
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$$F = \{f(A,B), f(B,C), f(A,D), f(B)\}$$

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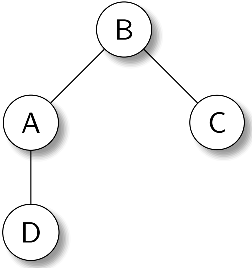

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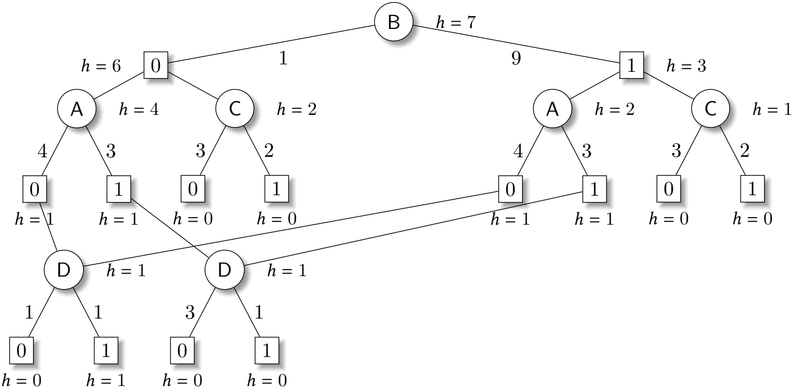

Algorithm Sketch

Down Pass: Expand nodes and mark terminal nodes solved

Up Pass: Update $v(n)$ for each node according to the following rules:

- OR Nodes: $v(n) = \min_{k \in \mathrm{succ}(n)} w(n,k) + v(k)$
- AND Nodes: $v(n) = \sum_{k \in \mathrm{succ}(n)} v(k)$

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