Weighted Heuristic Anytime Search
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Basic Heuristic Searches

- **Best-first Search**
  Blindly follows the heuristic

- **Weighted A* Search**
  For \( w > 1 \)

\[
f(n) = g(n) + w \cdot h(n)
\]

Larger \( w \) yields ‘greedier’ searches
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  For $w > 1$

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  Larger $w$ yields ‘greedier’ searches
Graphical Models

Definition (Graphical Model)

A tuple $\mathcal{M} = \langle X, D, F, \otimes \rangle$ where

1. $X = \{X_0, \ldots, X_{n-1}\}$ is a set of variables
2. $D = \{D_0, \ldots, D_{n-1}\}$ is a set of domains
3. $F = \{f_0(X_{S_0}), \ldots, f_{r-1}(X_{S_{r-1}})\}$ is a set of scopes:
   - $X_{S_i} \subseteq X$
   - $\forall i. f_i : X_{S_i} \rightarrow \mathbb{R}^+$
4. A combination operator $\otimes \in \{\Sigma, \Pi\}$

The model $\mathcal{M}$ represents the function

$$C(X) = \bigotimes_{i=0}^{r-1} f_i(X_{S_i})$$
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Graphical Models
Optimization Problems

Given a model $\mathcal{M} = \langle X, D, F, \otimes \rangle$, the most common optimization task is either *most probable explanation* or *maximum a posteriori*

**MPE** Find the optimal value $C^*$:

$$C^* = C(x^*) = \max_{X} \prod_{i=0}^{r-1} f_i(X_{S_i})$$

**MAP** Find the optimizing configuration $x^*$:

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Graphical Models
Optimization Problems: MPE/MAP → WCSP

**MPE**

\[ C^*_{\text{MPE}} = C(x^*) = \max_X \prod_{i=0}^{r-1} f_i(X_{S_i}) \]

**WCSP**  Weighted Constraint Satisfaction Problem
(MPE in negative log-space)

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Definition

The *primal graph* of a model is a graph where the vertices are the variables and edges connect variables within the same scope.

Scopes:
- $X_{S_0} = \{A, B\}$
- $X_{S_1} = \{A, C\}$
- $X_{S_2} = \{C, D\}$
- $X_{S_3} = \{B, D\}$
- $X_{S_4} = \{B, F\}$
- $X_{S_5} = \{E, F\}$

Figure: Primal
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*Figure: Primal*
**AND/OR Search Graphs**

**Pseudotrees**

(a) Primal

(b) Induced Graph

*Figure:* Induced graph over the natural ordering.
AND/OR Search Graphs

Pseudotrees

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Figure: Induced graph over the natural ordering.
AND/OR Search Graphs
Pseudotrees

(a) Orig + Ind. Edges
(b) Pseudo-tree

Figure: Pseudo-tree with edges chosen to respect the order
Pseudotrees

Figure: Pseudo-tree with edges chosen to respect the order
AND/OR Search Graphs

![Figure: Context-Minimal AND/OR Graph For Pseudotree](image)
Assume a graphical model $\mathcal{M} = \langle X, D, F, \otimes \rangle$ with primal graph $G$, pseudotree $\mathcal{T}$, and AND/OR search tree $S_T$.

**Definition**

The *context-minimal AND/OR search graph*, denoted $C_T$, is the AND/OR search graph obtained after merging all identical subproblems.

$C_T$ is exponential in the depth of $\mathcal{T}$. 
Assume a graphical model \( \mathcal{M} = \langle X, D, F, \otimes \rangle \) with primal graph \( G \), pseudotree \( T \), and AND/OR search tree \( S_T \).

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AND/OR Search Graphs
Optimization Problems

Assume a graphical model $\mathcal{M} = \langle X, D, F, \otimes \rangle$ with primal graph $G$, pseudotree $\mathcal{T}$, and AND/OR search tree $S_\mathcal{T}$

Definition

A solution tree $T$ of $C_\mathcal{T}$ is a subtree satisfying the following conditions:

1. It contains the root of $C_\mathcal{T}$
2. If an internal AND node $n$ is in $T$, then all the children of $n$ are in $T$
3. If an internal OR node $n$ is in $T$, then exactly one child of $n$ is in $T$
4. Every leaf in $T$ is a terminal node
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State-of-the-art $A^*$ for AND/OR search space.
Too complicated to fit on a slide

Highlights

Input:  
- Graphical Model $\mathcal{M} = \langle X, D, F, \Sigma \rangle$
- Initial weight $w_0$
- Pseudotree $T$ rooted at $X_1$
- heuristic $h_i$ (precalculated)

Output: Optimal solution to $\mathcal{M}$
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AND/OR Best First Search

Example

Let $\mathcal{M} = \{X, D, F, \Sigma\}$ where

- $X = \{A, B, C, D\}$
- $D = \bigcup_{s \in X} \{0, 1\}_s$
- $F$ is given by the following tables:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$f(A, B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B</th>
<th>$f(B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B</th>
<th>C</th>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
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AND/OR Best First Search

Example

Scopes

\[ F = \{ f(A, B), f(B, C), f(A, D), f(B) \} \]
AND/OR Best First Search Example

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Figure: Primal Graph
AND/OR Best First Search
Example
Algorithm Sketch

**Down Pass:** Expand nodes and mark terminal nodes solved

**Up Pass:** Update $\nu(n)$ for each node according to the following rules:

- **OR Nodes:**
  $$\nu(n) = \min_{k \in \text{succ}(n)} w(n, k) + \nu(k)$$

- **AND Nodes:**
  $$\nu(n) = \sum_{k \in \text{succ}(n)} \nu(k)$$
AND/OR Best First Search

Example

\[ \begin{array}{c}
A & B & C & D \\
\text{AND/OR Best First Search Example}
\end{array} \]