580 F11 2011-08-09

Note Title 2011-09-08

Lemma 1 (HNR-1968) (A^*)

For any optimal path P from s to n, there exists an open node n' on P with $\hat{g}(n') = g(n')$

[Notation: $\hat{g} \leftrightarrow g$, $g \leftrightarrow g^*$]

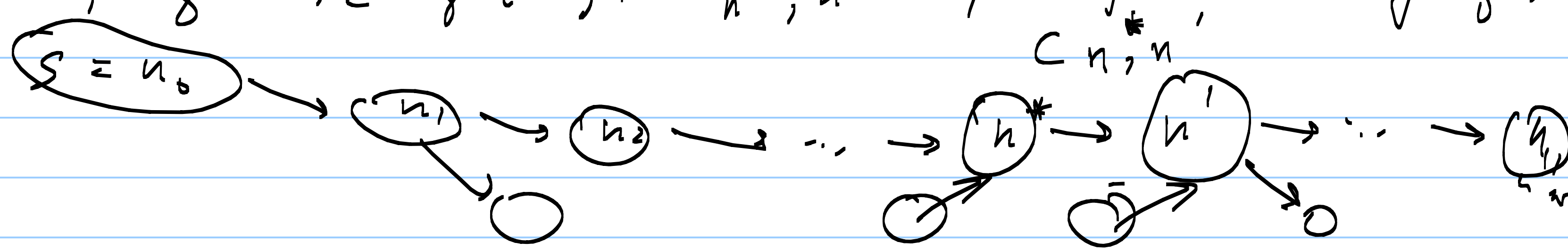
Proof

Let $P = \langle s = n_0, n_1, n_2, \ldots, n_k = n \rangle$.

If s is open (i.e. A^* has not complete even one iteration), then let $n' = s$, and the lemma is trivially true, since $\hat{g}(s) = 0 = g(s)$.

Let Δ be the set of all closed nodes n_i in P for which $\hat{g}(n_i) = g(n_i)$. This set Δ is non empty because it contains at least s.
Let n^* be the element of Δ with highest index.

Let n' be the successor of n^* in P.
Now, $\hat{g}(n') \le \hat{g}(n^*) + c_{n^*,n'}$ by definition of $\hat{g}$.



But since n^* is in Δ, $\hat{g}(n^*) = g(n^*)$
and since n' is on P, $g(n') = g(n^*) + c_{n^*, n'}$.
Therefore, $\hat{g}(n') \leq g(n')$. But
$$\hat{g}(n') \geq g(n'), \text{ so } \hat{g}(n') = g(n').$$
But since n^* is the highest indexed closed node for which $\hat{g}(n_i) = g(n_i)$, then n' is open. $\square$

(2) Theorem: A^* is admissible

Suppose A^* terminates with goal node t for which $f(t) = g(t) > C^*$. Then, per step 3 of A^*, $f(t) \le f(n)$ in OPEN just before termination, where n is a node on the optimal path, which A^* supposedly did not find.

But this contradicts Lemma 1.

(3) $f(n) \le C^*$ for all nodes expanded.

(A necessary condition for node expansion)

Proof: expanding a node for which $f(n) > C^*$ contradicts Lemma 1.

(4) Any node for which $f(n) < C^*$ will eventually be expanded by A^*.

(5) ... more informed ....

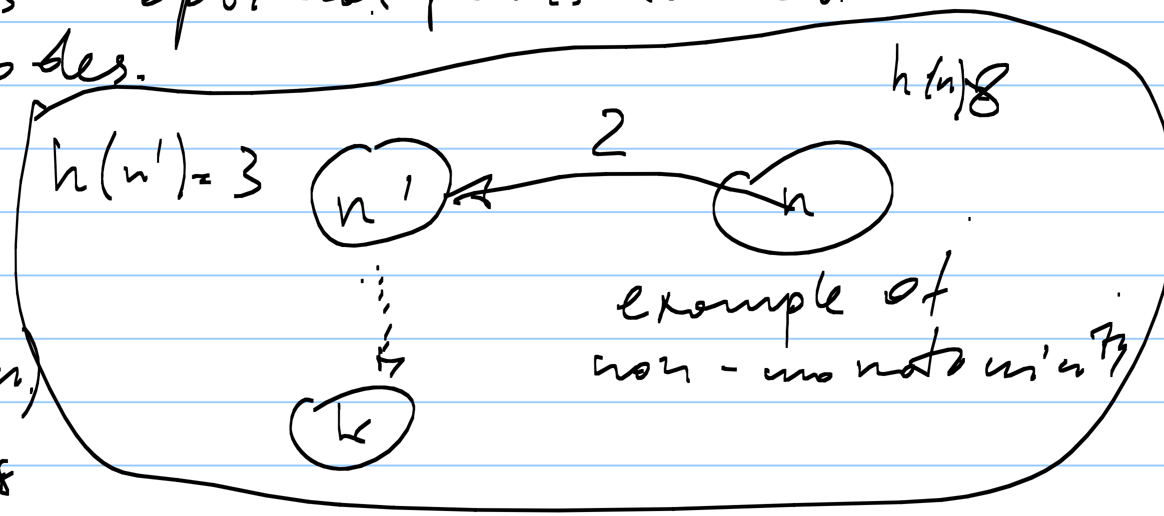
<u>Defn.</u> A heuristic h is <u>monotone</u> if $h(n) \le c(n, n') + h(n')$ for all (n, n'), where n' is a successor (child) of n

<u>Defn.</u> A heuristic is <u>consistent</u> if $h(n) \le c(n, n') + h(n')$, where n' is a descendant of n

monotonicity & consistency are equivalent

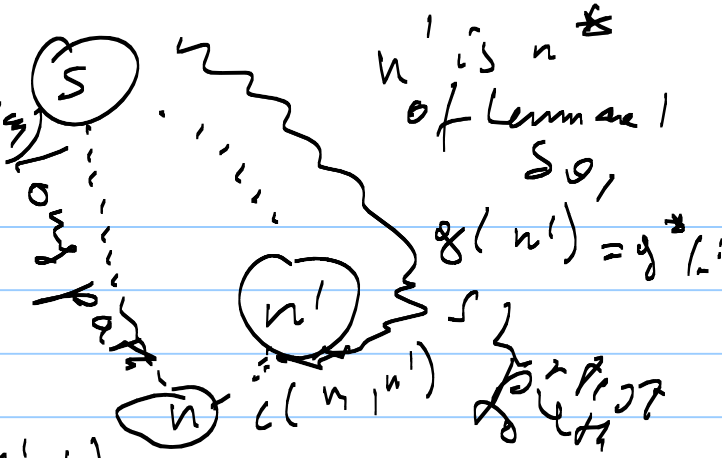
(6) An A^* algorithm guided by a monotone heuristic finds optimal paths to all expanded nodes.

(I.e., $g(n) = g^*(n)$ for all expanded nodes n.)



Assume that A^* expands a node for which $g(n) > g^*(n)$.

$f(n') = g^*(n') + h(n') \le$ (consistency)
$\le g^*(n') + c(n', n) + h(n)$



But, since n' is on the optimal path from s to n, $g(n) = g(n') + c(n', n)$

Therefore, $g(n) > g(n')$ implies $f(n') < f(n)$, and therefore n is not expanded by A^*. $\square$

(7) Monotonicity implies that the sequence of nodes expanded by A^* is non-decreasing.

Proof

Suppose n_2 is expanded immediately after n_1. If n_2 was on OPEN when n_1 was there, then obviously $f(n_1) \le f(n_2)$.

Otherwise, n_2 is a successor (child) of n_1. So,

$$g(n_2) = g(n_1) + c(n_1, n_2) \quad \text{and}$$

$$f(n_2) = g(n_2) + h(n_2) = g(n_1) + c(n_1, n_2) + h(n_2)$$

$$\ge g(n_1) + h(n_1) = f(n_1)$$

b/c $g\cancel{(n_1)} + c(n_1, n_2) + h(n_2) \ge g\cancel{(n_1)} + h(n_1)$

$h(n_1) \le c(n_1, n_2) + h(n_2)$ (monotonicity)

$\square$