Give a loop invariant for this program fragment:

\[ \begin{align*}
x &:= 2; \\
i &:= 1; \\
(* \text{What is the precondition here?} *) \\
\text{while } (i \leq n) \text{ do} \\
& \quad \text{begin} \\
& \qquad x := x \times x; \\
& \qquad i := i+1 \\
& \quad \text{end}
\end{align*} \]

with precondition \( n \geq 1 \) and postcondition \( x = 2^n \).

\textbf{Answer:} \( x = 2^{i-1} \land i \leq n + 1 \).

Also answer the following questions.

1. What is the precondition before the loop? \textbf{Answer:} \( x = 2 \land i = 1 \land n \geq 1 \)

2. Your invariant should consist of the conjunction of two formulae. The second formula is: \( i \leq n + 1 \). Why is this formula needed? \textbf{Answer:} To insure that \( i = n + 1 \) (rather than just \( i > n \)) when the loop is exited.

3. Show that the precondition at the line with asterisks implies the invariant. \textbf{Answer:} In short: (1) Since \( i = 1 \) and \( x = 2 \), then \( x = 2^{i-1} \). (2) Since \( i = 1 \) and \( n \geq 1 \), then \( i \leq n + 1 \).

4. Show that the invariant together with the negation of the loop implies the postcondition. \textbf{Answer:} In short: (1) Since \( i \leq n + 1 \) and \( i > n \) then \( i = n + 1 \). (2) Since \( x = 2^{i-1} \) and \( i = n + 1 \), then \( x = 2^n \).

5. Let \( x \) be the value of the variable \( x \) before executing the body of the loop and \( x' \) be the value of the variable \( x \) after executing the body of the loop. Write an equation that relates \( x \) and \( x' \). \textbf{Answer:} \( x' = x \times x \).

6. Let \( i \) be the value of the variable \( i \) before executing the body of the loop and \( i' \) be the value of the variable \( i \) after executing the body of the loop. Write an equation that relates \( i \) and \( i' \). \textbf{Answer:} \( i' = i + 1 \).