Induction & Prolog

Mathematical induction is a technique used to prove a property of the non-negative integers (the natural numbers).

Call the property $S(n)$. Math. ind. works this way:

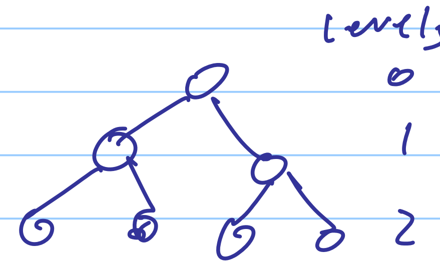
Prove $S(0)$ (base case)

Prove that, if $S(k)$ holds, then $S(k+1)$. (induction step)

$\to S(n) \to \dots \to S(2) \to S(1) \to S(0)$

Mathematical induction is a useful technique that is intuitively correct, but that is normally taken as a principle.

(One can show that it is "equivalent" to the least number principle.)



Example of proof by induction:

$$S(n) : \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Proof:

Base case: $S(0)$: $\underbrace{\sum_{i=0}^{0} 2^i}_{1} = 2^{0+1} - 1 = 2 - 1 = 1$

$\stackrel{!}{=} 2^0 = 1$

Induction step

Assume $S(k)$. Show $S(k+1)$

$$S(k+1): \sum_{i=0}^{k+1} 2^i = 2^{k+1+1} - 1 = 2^{k+2} - 1$$

We assume $S(k): \sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ (induction hypothesis)

$$\sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1} = (\text{use } S(k)) = 2^{k+1} - 1 + 2^{k+1} =$$

$$= 2(2^{k+1}) - 1 = 2^{k+2} - 1$$, so $S(k+1)$ holds ✓

Why does this matter for (Prolog) programming?

Consider the following claim about the above/2 predicate, above(X, Y):

$S(n)$: above(X, Y) if X is over Y with n blocks between them.

Proof. (by induction)

Basis: $S(0)$: above(X, Y) if X is over Y with $\emptyset$ blocks between them. The program has the clause

above(X,Y) :- on(X,Y). ✓

Induction step

Assume $S(k)$.

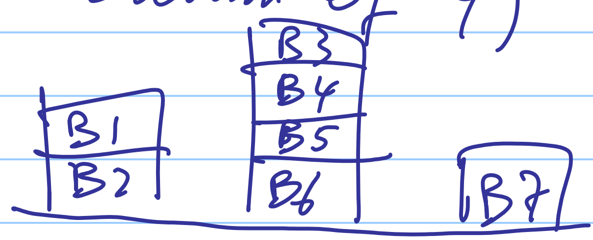
Show $S(k+1)$.

above(X,Y) :- on(X,Z), above(Z,Y).

$S(k)$ (above(Z,Y)) together with on(X,Z) imply above(X,Y). The inductive hypothesis holds because, if there are $k+1$ blocks between X and Y, there are k blocks between Z and Y.

Next example: the correct of left (X, Y).

$S(n)$: the predicate left(X, Y) is defined correctly when (the number of blocks between X and the table) + (the number of blocks between Y and the table) + (the number of block columns between the column of X and the column of Y) is n.



$S(3)$ could be left$(B1, B4)$

$3 = 1 + 2 + 0$

Proof (program construction) by induction;

$S(0)$: left(X,Y) :- just_left(X,Y).

Assume $S(k)$, i.e., left(X,Y) is defined properly for $n=k$. (Induction hypothesis)

There are three possibilities to consider for $S(k+1)$;

(1) ~~X~~ X is on top of a block

$$\underbrace{\text{left}(X,Y)}_{S(k+1)} \text{ :- on}(X, Z),\ \underbrace{\text{left}(Z,Y)}_{S(k)}$$

(2) Y is on top of a block

$$\underbrace{\text{left}(X,Y)}_{S(k+1)} \ \text{:-}\ \text{on}(Y,Z),\ \underbrace{\text{left}(X,Z)}_{S(k)}$$

(3) X and Y are on the table and X is to the left of Y

$$\underbrace{\text{left}(X,Y)}_{S(k+1)} \ \text{:-}\ \text{just-left}(X,Z),\ \underbrace{\text{left}(Z,Y)}_{S(k)}.$$

It is easier to argue that these 3 cases are exhaustive if you start from case (3);

(3) both X and Y are on the table
(1) X is on the table
(2) Y is on the table

Note that (1) and (2) are not exclusive.